THE

MATHEMATICAL GAZETTE

EDITED BY

T. A. A. BROADBENT, M.A.

62 Coleraine Road, Blackheath, London, S.E. 3

LONDON

G. BELL AND SONS, LTD., PORTUGAL STREET, KINGSWAY

Vol. XIX. December, 1935. No. 236

THE TEACHING OF MATHEMATICAL ANALYSIS IN SCHOOLS.*

By N. R. C. Dockeray.

The subject about which I have to speak to you to-day is one which was bound, in the course of time, to occupy the attention of the Mathematical Association. This Association was founded, as you know, for the purpose of improving geometrical teaching, but since its foundation, some sixty years ago, its activities have widened. It no longer confines itself to Elementary Geometry, but has issued reports on the teaching of Arithmetic, Algebra, and Mechanics. None of these, however, except possibly the last, touches on anything more advanced than School Certificate work, and even the Mechanics report does not deal with anything far in advance of this standard. The Association has, of course, done a great deal for the teacher of scholarship work through articles in the *Gazette*—indeed many of these articles are on subjects so advanced as to be quite out of reach of either scholarship candidates or their teachers, and to be of interest only to those who have attained a very high degree of specialisation. This is not a criticism ; it is entirely as it should be, for to my mind the function of this Association, which I believe is as yet too young either to realise its possibilities or to have attained its full strength, is to guide teachers of Mathematics throughout all stages of their pupils' careers. I believe that it would be a disastrous limitation of its activities if it were to confine itself to school mathematics only, and still more disastrous should it consider school mathematics as ceasing with the School Certificate.

Nevertheless the articles which have appeared in the *Gazette*, although many have been, and are, extremely helpful to teachers and to pupils, deal only with isolated points which arise in the normal scholarship course. A great deal of light, for example, has

* A paper and discussion at a meeting of the London Branch of the Mathematical Association, 23rd February, 1935.

been thrown on such subjects as " The use of differentials ", " How shall the exponential function be introduced ? " and other kindred matters. But so far no attempt has been made to coordinate these or to form any connected report dealing with the almost innumerable difficulties which confront the teacher of scholarship mathematics. The time for such a report has, I believe, now come. I say " a report ", although perhaps many of you will think that one report only will be insufficient. No doubt you would there be right, but I feel that scholarship mechanics may, for the time being, be well trusted to look after itself; and as for geometry, the splendid pioneer work done by one of the most active members of this Association will hold the geometric structure upright for some time to come. It is, however, true that, sooner or later, scholarship geometry must come under our survey, if only for the purpose of considering how best we may bolster up the somewhat rickety foundations on which projective geometry, as taught in schools, is built.

But at the moment it is Analysis that is most in need of help. And by Analysis I do not mean merely the differential and integral calculus, but the whole course of algebra, trigonometry and calculus which is to serve as the basis upon which the university lecturer hopes to build those two tremendous structures, the theory of functions of a real variable and the theory of functions of a complex variable.

Let me say here at the outset that you are going to get very little help from me this afternoon. Mr. Daltry asked me some time ago to read a paper on this subject, giving as the ground for his choice of speaker the fact that I had written a book on Analysis. It struck me at the time that this was a very inadequate reason: how completely inadequate it was, however, I did not fully realise until I started to think out what I was going to say. Then it was that my lack of qualifications for this task became fully apparent, and again and again I found myself quailing before the task of setting anything on paper. In the last resort sheer desperation, induced by the imminence of the day on which I was to speak, overcame my timidity and my natural indolence, and the result is now being presented to you with very considerable diffidence, a diffidence which has been enormously increased by Professor Neville's address last January. I shall have occasion to refer to that address again later in this discourse.

Now let us consider for a moment what *are* the difficulties which confront the teacher of scholarship analysis. In one direction they are very similar to those encountered in teaching elementary geometry. I refer to the problem of deciding how much we shall assume as obvious. In analysis, as in geometry, the most fundamental things of all present enormous difficulties, and often to the student appear to be such platitudes that it is a waste of time trying to prove them. As an example I might cite the concept of the irrational number. It is not difficult to convince even a school certifi-

cate candidate that there is something queer about the square root of 2; in fact, he is frankly incredulous when it is pointed out that when expressed as a decimal it neither terminates nor recurs, as he does not believe that such a thing can possibly happen. So strong is this conviction that even if you present him with a decimal which quite obviously neither terminates nor recurs, such as

$$0{\cdot}1234567891011\ldots,$$

in which the natural numbers are written in order following the decimal point, the boy is extremely sceptical about it and is inclined to think that he is being hoodwinked in some such way as in the "proofs" that $1=2$ or that all triangles are equilateral. In this connection it is as well to remark that the average boy has no very profound faith in mathematics, and is apt to have a vague distrust of figures, which, as he knows well, can be made to prove anything. Still he is usually a little taken aback when it is proved to him that $\sqrt{2}$ is a non-terminating non-recurring decimal. It is probably his faith in himself that is shaken rather than anything else, but that is irrelevant to the point I am now emphasising, which is that we can easily persuade even a boy of fifteen that $\sqrt{2}$ is "queer". On the other hand, it is extremely difficult to persuade even the scholarship candidate that this queerness deserves more than passing notice—it is as difficult to make him realise that irrational numbers and operations upon them require adequate definition as it is to make his younger brother realise that the opposite sides of a parallelogram must be *proved* equal. This is one example of the type of difficulty to which I am here referring, but many others could be chosen. Continuity, Rolle's theorem, the fact that a continuous function attains its bounds, or that $\frac{\partial^2 z}{\partial x \partial y}=\frac{\partial^2 z}{\partial y \partial x}$ when these are both continuous, all these present difficulties of a similar nature.

And the method we must use in overcoming them is parallel to that adopted in dealing with elementary geometry. During the first year we must assume anything which the student would naturally regard as obvious. This course has the double advantage of keeping the pupils interested and of enabling the teacher to cover a large amount of ground. It must be remembered that the initial stages of any new subject should be made as interesting as possible, since the damage done by a dull presentation at the beginning is incurable. Later on you may bore your pupils as much as you like, they will simply go to sleep and their mental attitude will remain unimpaired, but let the opening stages be full of interest.

I suggest then that an attempt should be made to cover the following course during the first year of specialisation. I put this forward with no suggestion of finality, it is to be regarded as the expression of views which I hold only tentatively, and I am open to conviction on almost any point which may be raised in connection with it.

We start the year with an introduction to integration as the limit of a sum and deal with simple cases, such as x^2, x^3, $\sin x$ and $\cos x$.

This can be at once applied to the calculation of simple areas. We next inquire if it is possible to extend the method to deal with more difficult cases. We show that if

$$\int f(x)\,dx=\phi(x)$$

then

$$\lim_{h\to 0}\frac{\phi(x+h)-\phi(x)}{h}=f(x).$$

The class will by this time be ready to agree that limits are going to occupy a central position in this new work. We therefore have an interlude here to consider limits. These should be approached via infinite sequences, but as the discussion on this point at the January meeting will appear later in the *Gazette*, I shall not enlarge on it now. When a reasonably firm grasp of limits has been gained we are ready to attack the differential calculus. By adopting this order the differential calculus acquires in the students' minds a *raison d'être* which it is apt to lack if taken before the integral calculus. Maxima and Minima are a limited field, and the construction of tangents to curves appears to the average boy to have less solidity than the calculation of areas. Moreover, there is no need to stop at areas; volumes of solids of revolution and even simple cases of centre of mass are within his grasp in a very short time. Surfaces are more difficult and should be avoided at this stage. The value of the formula for the differentiation of a function of a function, so often a stumbling-block when the differential calculus is treated first, is borne in on the student without difficulty when the emphasis is being laid on the integral calculus, in fact he finds himself using it almost before he realises it. When the utility of integration has been firmly driven home and the class begins to realise the power of this new instrument, we can return to the differential calculus and make fairly rapid progress. The derivative as a rate of change, and maxima and minima should be dealt with next. Also it is advisable to introduce functions of two variables and Taylor's theorem as early as possible. Of course the treatment must be as simple as possible, and geometrical illustration should be freely used. But I regard it as important that some facility in dealing with partial derivatives should be acquired during the first year's course. The same applies to Taylor's theorem: as soon as the students have acquired a reasonable degree of skill in differentiation it should be introduced; some sort of reasoning, however specious, may be used to back it up, and the students may be allowed to play with the theorem more or less as they please; for instance, to work out expansions without bothering much about convergence or remainders. By this means it is possible to fascinate them with the power of the calculus, and to create a genuine enthusiasm which one finds to be largely absent in those who have followed a more sedate course. Needless to say, I do not for a moment suggest that the pupils should be allowed to suppose that they are being given

valid proofs. On the contrary, it should be constantly impressed on them that they are merely being permitted to enjoy themselves for a year, and that the whole course must be covered again, rigorous proofs being substituted for the toy proofs they are now receiving. During this year it should be possible also to deal with partial fractions, easy determinants, de Moivre's theorem, permutations, the simpler cases of convergence (let us say the comparison tests and d'Alembert's test), and the binomial theorem. By this time the boy will be a little tired of forging ahead so rapidly and will be well content to rest on his oars while he contemplates with a rather more careful scrutiny what he has learnt. He is therefore in a proper frame of mind to enter on his second year's course.

And it is here that the real obstacles are to be found. One of these, and that the most difficult to overcome, arises in connection with the organisation of our classes and time tables : it is really an economic problem, and I will postpone consideration of it until later. The others may be summarised in the question : " What shall we teach, and how shall we teach it ? " In answer to the first part of this inquiry, I have written on the board a syllabus * which I think is certainly sufficient, and which I hope will some day be considered necessary. The syllabus is set out in the order in which I think it is advisable to take it. It is the second part of the question that I now propose to discuss.

I regard it as important during this second year of scholarship work that every effort should be made to be as rigorous as is compatible with the abilities of one's pupils. The transition from school to university constitutes a definite break, and it is vital that the student should not have to unlearn anything when he reaches the university. Everything he learns during this year should therefore be presented as far as possible in logical order and with an unassailable backing. There are, however, a few items which will almost certainly have to be postponed until the end of the school course, and may possibly have to be omitted altogether. They are

(1) The idea of sections and Weierstrass' theorem on points of accumulation ;
(2) The Heine-Borel theorem ;
(3) The theory of Riemann integration.

I shall make a few brief remarks about each of these before passing to the consideration of details arising in that part of the work which all will agree to be essential.

(1) The concept of sections of rational numbers, of course, lies at the foundation of the whole of analysis. Nevertheless I think it should not be forced at this stage. Many may disagree with me here, and it is true that a number of proofs can be shortened by

* *Elementary Treatise on Pure Mathematics*, by N. R. C. Dockeray. Ch. I (omitting Art. 35), II, III, IV, V, IX, VI. Ch. X, Arts. 179-196, 201-206. Ch. XI. Ch. XII, Arts. 226-228, 235-237, and 246-253. Ch. VII. The rest of Ch. X and XII, Art 287.

Pure Mathematics, by G. H. Hardy. As much of this as can possibly be fitted in.

their use, but the fact remains that boys find them beyond their grasp. Perhaps this is not quite a fair statement of the difficulty—the *idea* of sections is simple enough, it is the awe-inspiring baldness of the theorems and concepts dependent on them, points of accumulation, limits of indetermination, and the like, that is so chilling to the enthusiasm. All these should if possible be dealt with before the end of the school course, when the student has realised the value of the ideas involved, but the beginning of the second year is not the time for them. Meanwhile the difficulty can be dodged by assuming frankly that a bounded monotonic sequence possesses a limit, though it should be pointed out when making this assumption that it is the irrational number which causes the trouble.

(2) The chief result depending on the Heine-Borel theorem is the theorem that a function which is continuous in a closed interval is uniformly continuous in this interval. If we make the assumption on monotonic sequences mentioned above, this theorem can be proved by the method of continued bisection quite easily and certainly much more intelligibly than by first stating and proving the Heine-Borel theorem in all its cold nudity and then quoting it. Moreover, the method of continued bisection will probably be already familiar to the students, since in the absence of the theory of Dedekind sections it will almost certainly have been used to establish such theorems as "a function which is continuous in a closed interval is bounded in this interval", "if $f(x)$ is continuous in the interval (a, b) and if $f(a)$ and $f(b)$ have opposite signs, then $f(x)$ vanishes for at least one value of x between a and b". The method is moreover both simple and convincing.

(3) The theory of the definite integral will probably be regarded as the most important of the three points I have raised. Nevertheless it is improbable that, as things are at present, it will be possible to find time to establish the existence of the Riemann integral satisfactorily, but in this case it should be made clear to the students that a definite assumption is being made. I might point out that in the case where a continuous function is to be integrated over an interval which can be divided into a finite number of parts in each of which the function is monotonic, the existence of the integral is very easy to establish; and if time for the more general case is lacking, this at least should be substituted, some indication being given of the possible existence of cases left uncovered.

I must now deal with a few details arising in that part of the scholarship course which will generally be admitted to be essential. A very large number of problems here present themselves for consideration, but I shall confine myself to half a dozen of the more salient points.

(1) Students should be quite clear as to what classes of functions they are capable of integrating. In particular, one of the three general methods of dealing with rational functions of x and

$$\sqrt{(ax^2+2hx+b)}$$

should be firmly implanted. If possible they should also learn that if y is defined as a function of x by an equation $\phi(x, y)=0$, and if the curve represented by this equation is of genus zero, then any rational function of x and y can be integrated with respect to x in terms of elementary functions only. Of course a proof of this should not be attempted, but the fact itself is of considerable importance, and should not be omitted.

(2) As much practice as possible should be given in working with double integrals and with functions of two variables. These are of such vast importance in all branches of applied mathematics that they cannot be too greatly emphasised. Probably we will have to content ourselves with a bare outline of the theory of both these subjects, but that is a matter for individual decision and need not detain us here.

(3) The fact that every algebraic equation has a root is one which is commonly left unproved, or, what is worse, unstated. In other words, students are left to suppose that they are making no unjustified assumption when they use the theorem, as they do every day. This should not be so. Even if the theorem is not proved it should at least be pointed out that it is one which requires proof. But, in fact, it should be well within the grasp of most scholarship candidates, though it is best left until somewhere near the end of their course. The method of proof given in Chrystal's *Algebra* (vol. i, p. 248) is of course quite useless as it stands, but it is quite sound if supplemented by a proof that a continuous function of two variables attains its bounds. This unfortunately is probably the more difficult half of the proof, but as it requires nothing more than an extension to two dimensions of the method used in proving the corresponding theorem for the case of one variable, it should not be beyond the power of most scholarship candidates, if time can be found for it. Alternatively the proof given in Hardy's *Pure Mathematics* can be used.

Another theorem which is in daily use and which frequently escapes attention is that if an infinite power series in x vanishes for all values of x for which it is convergent, then every coefficient vanishes separately. The simplest proof of this theorem depends on the theory of uniform convergence, which, of course, is outside the scope of scholarship work. The proofs given by Charles Smith, and by Hall and Knight, assume uniform convergence and would appear to have little value as they stand. Chrystal first proves the continuity of the infinite series and then gives the same proof as is found in Smith: the argument is thus complete but too lengthy to be easily assimilated. A straightforward algebraical proof which is neither long nor difficult will be found in Bromwich's *Infinite Series*, Art. 52. I have reproduced this proof in Art. 287 of my own book.

(4) During the first year's course both the Binomial and Exponential series will probably have been met with and treated either on the lines indicated in Durell's *Advanced Algebra* or by an application of Taylor's theorem. But the latter, of course, will not at

this time have been formally proved. In the second year's course, therefore, it will be essential to place both these theorems upon a satisfactory basis. Now the deduction of the binomial theorem from Taylor's theorem is evidently one of the best methods that can be given, since it affords a proof which applies equally well to an irrational as to a rational index. Unfortunately this presupposes that the meaning of an irrational index has been defined, and for this we must appeal either to the exponential theorem or else to the idea of sections, whereas in fact the binomial theorem will probably precede both of these. Of course there is no reason why we should not use Taylor's theorem to establish the binomial theorem first in the case when the index is rational, and then, later, when the exponential theorem has been dealt with and the meaning of an irrational index defined, pointing out that the same method of proof now applies to the more general case. But this order of procedure is open to objections, chief among which is the fact that the students are using both the binomial and exponential theorems long before they are given a sound proof of Taylor's theorem. This upsets their faith considerably and is in any case an offence against the principle, which I believe to be vital, that during the last year of preparation for scholarships the course offered should be not only rigorous but also consecutive. We should make every effort to avoid saying "We require this result now, but we shall not be in a position to prove it until next term". Let me repeat here that when the scholar leaves school he should have nothing to unlearn, but should be equipped with a sound knowledge of the elements of the Theory of Functions arranged in their proper sequence, so that the university lecturer has secure foundations on which to build. It must be remembered that at the university the students are not subject to the almost parental guidance which they receive at school, and that gaps in their knowledge may be left unfilled, and hazy ideas about logical sequence may be left unclarified, throughout their whole university career. It behoves us therefore to see that they are not left in such a perilous plight.

Returning then, there have been of late years rather a spate of new proofs of the binomial theorem, chiefly based on uncouth integrations. On examination these will be found in the main to consist of applications of the proof of Taylor's theorem by definite integration to the particular function $(1+x)^m$. As such they are open to the objections I have outlined above and in addition they naïvely assume the first mean value theorem for integrals or some other theorem indistinguishable from this. Now, why all this fuss? The binomial theorem for a rational index is a simple theorem of algebra, and, as Professor Hardy has reminded us, it ought, as such, to possess an algebraic proof. Of course we all know the elementary proof by the multiplication of series, but for some reason which I have never succeeded in grasping, this method is considered too old-fashioned for the modern class-room. It is true that it must be preceded by the product theorem for absolutely convergent series,

but I maintain that this ought to be dealt with at the beginning of the second year's course. Even, however, if this turns out to be impracticable I would prefer to assume this theorem than to use any of the methods depending on integration. It is true also that Vandermond's theorem is required, but if a boy cannot understand Vandermond's theorem he should not be working for a scholarship. The method which is commonly known as Euler's proof should be given a wide berth—to me it would appear to carry no validity whatsoever, but I am open to conviction on this point. However, as I have remarked previously, a comprehensive proof for all real indices must ultimately be given by means of Taylor's theorem, but this should fall into its proper place in the school course.

I believe also that the case of the complex index ought not to be neglected. Various methods of proof may be used, by far the most elegant of which is that given by Cauchy. This method is so neat that I would be prepared to give it with the necessary cautionary rider attached, but as it rests on a knowledge of the theory of double series, which is necessarily well outside the school syllabus, it should not be allowed to constitute the only support of the general binomial theorem. By extending Taylor's theorem to cover the case of complex functions of a real variable the theorem can be made to rest on a firmer basis. This method is fully set forth in Hardy's *Pure Mathematics*.

What I have said about the binomial theorem applies also, *mutatis mutandis*, to the case of the exponential theorem. This theorem will almost certainly have been met with during the first year's course, and I think it is of little importance how it has been introduced, except that the treatment should be as simple as possible. Probably a juvenile development from the equation $\frac{dy}{dx}=y$ is the most satisfactory. But during the second year, the year of facts and logic, the algebraic method depending on the multiplication of series is by far the best and simplest in the long run. Moreover, it enables the theorem to be taken in its proper place.

(5) Taylor's theorem evokes little comment. The two standard methods of proof should both be given. I myself find that the proof by definite integration carries more conviction than that based on the ordinary mean value theorem. By conviction here I do not mean merely intellectual conviction. The proof of Taylor's theorem by applying the mean value theorem to that somewhat unwieldy function

$$f(a+h)-f(x)-(a+h-x)f'(x)-\ldots-\frac{(a+h-x)^{n-1}}{(n-1)!}f^{(n-1)}(x)$$
$$-\left(\frac{a+h-x}{h}\right)^{n}\left\{f(a+h)-f(a)-hf'(a)-\ldots-\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)\right\}$$

is so simple that no intelligent schoolboy can fail to follow each step and to admit that the result obtained is correct. But mathematics

is an art, not merely an intellectual exercise, and therefore in a mathematical proof something more than power over the intellect is required. There must be emotional conviction as well. This does not mean simply that the proof must satisfy certain aesthetic requirements, though this is undoubtedly an advantage. It must be such that the learner can " feel in his bones " that the result is inevitably true. I am not certain that in laying stress on this point I am drawing a real distinction at all—those who are better acquainted with the principles of aesthetics than I am may well assert that if a proof is satisfactory on aesthetic grounds it cannot fail to carry emotional conviction. This may be so, but it can be argued that the mean value proof of Taylor's theorem is in some respects more in accordance with aesthetic requirements than the other, since it does not involve the continuity of the last derivative. Yet it is undoubtedly a fact that the proof by definite integration is the more convincing of the two. But I am aware that I am treading on thin ice here, so I will venture no further in this direction. I should like, however, to drive home the distinction between intellectual and emotional conviction by recounting that on the last occasion on which I had to lecture on Taylor's theorem I set forth the usual mean value proof, culminating in the well-known result:

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\dots$$
$$+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)+\frac{h^n}{n!}f^{(n)}(a+\theta h),$$

where $0<\theta<1$.

The proof appeared to have been swallowed and digested well enough, but a moment later I had occasion to replace h by x. I was immediately held up by one of my pupils, who remarked, somewhat indignantly I thought, " But, Sir, you can't do that, you have only proved the theorem for *constants* ".

Before leaving Taylor's theorem I should like to issue one further warning. Students should not be allowed to suppose that the convergence of the series is all that is required to ensure its being a correct representation of the function from which it is derived. The actual remainder must always be examined. And as example is better than precept, I make a point of driving this home by citing Cauchy's function, which is defined by the equations

$$f(x)=\exp\left(-\frac{1}{x^2}\right) \text{ when } x\neq 0, \quad f(0)=0.$$

This function leads to the remarkable expansion

$$0+0\,.\,x+0\,.\,\frac{x^2}{2!}+0\,.\,\frac{x^3}{3!}+\dots,$$

which is evidently convergent, but bears little resemblance to the function from which it is obtained.

(6) I have left until the last what is probably the most tricky point of all. I refer to

$$\lim_{x \to 0} \frac{\sin x}{x}.$$

The ordinary procedure is roughly as follows. First $\sin \theta$, $\cos \theta$, etc., are defined geometrically as functions of an angle, the idea of angle itself and more particularly of its measurement being left to the intuition. Later the radian is introduced, usually defined as the angle subtended at the centre of a circle by an arc equal to the radius. The length of a curved line is, of course, here dealt with on intuitive lines. The area of a sector is then obtained, recourse being had once again to intuition; from this we get

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

by the well-known geometrical method, the derivatives of $\sin x$ and $\cos x$ follow and all seems to be well. We think no further about it until we come to deal rigorously with areas and lengths of arcs. We then learn, often with something of a shock, that these both require careful definition. So all our faith in $\lim_{x \to 0} \sin x/x$ was misplaced, the house has been built upon a quicksand! "There is no certainty in anything", we say—and a period of dejection supervenes. Soon, however, a ray of hope appears, our despondency begins to evaporate. "Why", we exclaim, "now that we know all about areas and arcs, why shouldn't we now *prove* that the area of a sector is $\frac{1}{2}r^2\theta$? Of course, we mustn't assume the limit of $\sin x/x$ in doing it—that would be *too* obvious an error—but surely we can avoid that." We soon find, however, that we can not avoid it; indeed, the more we contemplate this circle, whose area is so baffling, the more vicious it appears. Is there then no escape from this difficulty? Well, of course, there are several. One method is to define $\sin x$ and $\cos x$ by means of the infinite series, from this the addition theorems can be immediately deduced and also the derivatives of the circular functions. We must next establish their periodicity, and this is accomplished fairly easily by proving that the equation $\cos x = 0$ has one and only one root lying between 0 and 2. If this root is called $\frac{1}{2}\pi$ it at once follows from the addition theorems that $\cos(x + 2\pi) = \cos x$ and that $\sin(x + 2\pi) = \sin x$. Thus the whole of analytical trigonometry follows without further difficulty. It remains now to connect these definitions with the elementary ideas derived from right-angled triangles. To this end we observe that (since $\cos^2\theta + \sin^2\theta = 1$) the circle $x^2 + y^2 = a^2$ can be represented parametrically by the equations $x = a\cos\theta$, $y = a\sin\theta$. By using the formula $\frac{1}{2}\int(x\,dy - y\,dx)$ we can then prove that the area of the sector bounded by the lines $y = 0$, $y = x\tan\alpha$ and the arc of the circle from $\theta = 0$ to $\theta = \alpha$ is $\frac{1}{2}r\alpha^2$. Now, using any form of words

we please to give a general idea of what is meant by an angle, we define the measure of the angle between the lines $y=0$ and $y=x\tan\alpha$ to be α, and the whole geometric structure follows at once.

An alternative method is to define arc tan x by the equation

$$\theta=\text{arc}\tan x=\int_0^x \frac{dm}{1+m^2}.$$

It is not difficult then to prove that

$$\text{arc}\tan x+\text{arc}\tan y=\text{arc}\tan\frac{x+y}{1-xy},$$

provided that $xy<1$, that as x tends to infinity arc tan x approaches a finite limit which we agree to call $\frac{1}{2}\pi$, and that $\tan\frac{1}{4}\pi=1$. With this as our starting point the values of θ naturally confine themselves between $\pm\frac{1}{2}\pi$, so we define sec θ as being the positive square root of $1+\tan^2\theta$, and the remaining functions by the usual formulae. We are now in a position to prove that $\cos 0=1$, $\cos\frac{1}{2}\pi=0$, and that if θ, ϕ and $\theta+\phi$ all lie between $\pm\frac{1}{2}\pi$, then

$$\cos(\theta+\phi)=\cos\theta\cos\phi-\sin\theta\sin\phi.$$

By means of the addition theorems we can now extend the definitions of these functions to cover the cases where the argument is greater than $\frac{1}{2}\pi$. We are at once led to the periodicity properties and the rest of the argument follows as before.

Of these two methods it would seem that the first is the better, as the functions are defined at once for all values of the argument, and no extensions of the definitions are required. This method is set out completely in the appendix to Whittaker and Watson's *Modern Analysis*. But in practice there are objections to both. It is quite evident that in actual fact the trigonometrical properties of the functions are bound to be considered first and to be deeply ingrained in the minds of our pupils long before they learn that

$$\lim_{x\to 0}\frac{\sin x}{x}=1.$$

Now boys are conservative animals ; they dislike being told that they have started from the wrong end, and they derive much more satisfaction from a method of dealing with the problem which follows the order in which the subject has developed in their minds than from one which adopts precisely the reverse order. They are apt to regard the latter as an amusing illustration of the remarkable consistency of mathematics than as the essentially logical way of treating the subject. It is of course impossible to satisfy their requirements in this direction, partly on account of the difficulty inherent in the measurement of areas which I have already mentioned, but also because it is contrary to the whole spirit of analysis to base its reasoning on purely geometrical considerations. The

addition theorems, in particular, carry no analytical validity whatever as ordinarily proved, for not only do they rest on a geometric argument, but this argument itself is one which arouses considerable suspicion, involving, as it does, the idea of the equality of angles by superposition. And, of course, even if the limit of $\sin x/x$ is granted, we still require the addition theorems in order to differentiate $\sin x$, if we follow the conventional lines. In short, it will be agreed that this procedure is bristling with difficulties throughout. Ultimately we must adopt one or other of the analytical methods given above, or some other method akin to these; but for a first treatment we can circumvent at least some of the difficulties as follows. If we admit that, whatever definition is ultimately to be given to the concept of an area, it must in any case satisfy the following conditions:

(i) the areas of two congruent figures are equal;

(ii) if two areas are such that no point of one lies outside the other, then the first area is less than or equal to the second;

then we can adopt the method given by Euclid in Book XII, Prop. 2, to prove that the areas of the sectors of two concentric circles, which subtend the same angle at their common centre, are proportional to the squares on the radii of the circles. Thus if z is the area of such a sector in a circle of radius r, z/r^2 is independent of r, and therefore depends only on the angle at the centre. We define the measure of this angle to be $2z/r^2$, and, this done, the limit of $\sin \theta/\theta$ follows at once. But of course the other difficulties remain.

The course which I have outlined may seem to some of you to be a very comprehensive one, and you may well ask if it is possible to cover such a wide ground in the limited time at our disposal. I admit the difficulty, and I am prepared to confess that I make no pretence of practising my own precepts. The usual state of affairs is that after a year in a Mathematical Vth form, boys spend 15 (sometimes 27) months in a VIth form before they sit for their scholarship examination, and, to state the facts baldly, it would be a sheer impossibility, even with an exceptionally brilliant class, to work through the syllabus which in my opinion we ought to complete before sending our boys up to the university. It is indeed possible to cover a great part of this course, but time is so limited that there seems to be no opportunity to consolidate what we teach, to drive home the logical sequence of the course and to create in the minds of our pupils a clear perspective of the elements of analysis as a whole. And I maintain that this is a most important part of our work; moreover, it can never be attained by working examples, or indeed by any work which the students do alone, it can be reached only by a course of lectures. The condition of the mind of the average scholarship candidate when he leaves school is almost chaotic. He has acquired a vast and jumbled amount of knowledge, but he has practically no sense of direction, and is apt to

suppose that during his university career he is simply going to learn a whole crowd of new facts and new processes for solving problems. Well, of course, so he is, but these are not the only things he will learn. He must also fit these new facts and processes into their proper places so that they form a coherent whole, and it is at school that he must be taught how to do this. Perhaps I may make my meaning clearer by taking an actual example. The theory of Functions of a Complex Variable is, I suppose, one of the most shapely structures in the whole of mathematics. But if we were to present this subject to our pupils without impressing on them the relative importance of its various parts, if, for example, we were to lay as much stress on the evaluation of definite integrals by the theory of residues as upon Mittag Leffler's theorem or Weierstrass' factor theorem, seventy-five per cent. of the beauty and simplicity of the theory, and seventy-five per cent. of its educational value would be lost. The evaluation of definite integrals by residues is no doubt both important and interesting—it is indeed one of the most fascinating applications of Cauchy's theorem—but it does not form an integral part of the general function theory, whereas the other two theorems do. It is this sense of direction, of visualising the part as a component of the whole, that is lacking in our teaching of school mathematics. Leibniz' theorem jostles with Taylor's for pride of place in our pupils' minds, and both are in danger of being dislodged by some formula for the curvature. Recurring series, which, though both interesting and important, are nevertheless only an application of the binomial theorem, are given the same weight as the theory of absolute convergence. And other instances could easily be given. It is not the fault of the teachers—all these things have to be learnt, many hours must be spent on examples in order that our pupils may acquire facility, and we are usually thankful enough if we succeed in covering most of the scholarship course in the time at our disposal without hankering after the impossible ideal of consolidating this work by a course of lectures on the architecture of our subject. And yet we *must*, somehow or other, attain this ideal. I have already said that the university lecturer requires a solid foundation on which to base his work. But as things stand at present he doesn't get it. Indeed I have often wondered if directors of studies at the universities sometimes exclaim, with that impatience of despair which we so often display towards the preparatory schoolmaster: "Do they *ever* teach the boys *anything* at school?" May I remind you that it is now twenty-seven years since Hardy's *Pure Mathematics* was first published, and yet, so far as I can see, we are still as far as ever from attaining the ideals which are embodied in that book. Now Professor Neville's address to this Association last month makes it all too abundantly clear that he expects from us all that I have adumbrated and more. He requires, and in my view rightly requires, not only that the boys who come to him from the schools should have a sound knowledge of the elements of mathematical analysis with a clear perspective

of its various parts, but that in addition they should be *au fait* with the most modern methods. How are we to reach this seemingly inaccessible ideal ? Let us examine the difficulties in detail. They are twofold.

In the first place I think it would not be altogether untrue to say that we, the teachers in the schools, are very imperfectly equipped for the task. An architect cannot plan the foundations of a building if he has no idea, or only a vague idea, what form the superstructure is to take. But I have often felt myself to be in such a position. Even if I confine myself to analysis, I am bound to admit that there is a great deal of the subject about which I know nothing at all. In fact, except for the small field occupied by the scholarship course itself, my knowledge is so rusty that it might well be considered valueless. And I imagine that many teachers of scholarship mathematics would be prepared to make a similar confession. How, then, are we to guide our pupils if we cannot see the way clearly ourselves ? I shall make no attempt to answer this question.

The second difficulty seems to rest on a defect which is inherent in the whole educational system, and, so far as I can see, it can be overcome only by a reorganisation of the whole system. Perhaps I shall be able to explain the matter most clearly by recounting my own experience. I do not know to what extent it is typical, but I imagine that the position is not very different in other schools. Of the time which is allotted to me for teaching, sixty-four per cent. is spent with classes of certificate or pre-certificate standard, and the remaining thirty-six per cent. is devoted to work with the mathematical VIth, which is shared between one of my colleagues and myself. To be precise, my colleague takes the form ten periods a week and I take it nine. It would seem that to expect more than nineteen periods a week would be sheer greed, and so it would, if those periods were really available for teaching. But unfortunately the mathematical VIth is divided into four groups, no two of which are doing the same work, and as these groups all require approximately the same amount of attention, the effect is that each receives on the average $2\frac{1}{4}$ periods or about two hours of teaching per week. My colleague is in a similar position ; the net result is therefore that the boys who are to sit for scholarships next December are now receiving *less than five hours' teaching per week.* As a great part of these five hours must necessarily be devoted to going through examples, it will be realised that the time available is totally inadequate. And this is not an isolated state of affairs peculiar to this particular year ; my experience has been very similar to this ever since I started teaching. In actual fact the picture is even blacker than I have drawn it, but I have said enough to show that the position is, to put it mildly, laughably absurd. And I imagine that it is equally absurd in other schools. Of course the schools themselves cannot avoid this. It is manifestly impossible to separate these four groups into four separate classes and to put a master in

charge of each. No school can afford to employ four highly qualified men each of whom is to spend a great part of his time in taking a class of two boys. And yet the present system will never produce competent mathematicians. The solution would therefore seem to lie in a reorganisation of the whole system of post-certificate work, and I tentatively suggest that the lines of this reorganisation should be somewhat as follows. The whole country should be divided into ten or a dozen areas. In each area the various schools, public and secondary, should be subdivided, so far as post-certificate work is concerned, into about half a dozen groups, each group making a speciality of one particular subject. Thus schools A, B and C would specialise in mathematics, schools D, E and F in classics, and so on. A boy who had matriculated and who wished to sit for a scholarship in one subject would then be automatically transferred to one of the schools in his area in which that particular subject was taught. By this means a school which specialised in, say, mathematics, would find itself with so large a number of prospective scholars that it could, without financial loss, arrange for these scholars to be taught in classes of about eight or ten boys, all of whom would be doing the same work. Thus the engineers would form one class, the actuaries a second, the mathematical scholars a third, and so on.

I realise of course that there are a great number of objections which might be raised. Of these one of the strongest is the sentimental value which attaches to belonging to one particular school, though to my mind the eradication of this false patriotism is one of the great points in its favour, but somehow or other I do not see the British Parent agreeing with me here. Still even the British Parent is probably not much more difficult to educate than his son, and I think that the idea is worth considering. No doubt there are other schemes which would meet the situation equally well, indeed I have myself in mind a solution on quite different lines. But this solution is so revolutionary, so expensive, and so obviously right that even my privileged position as an Irishman, which entitles me to make surprising suggestions without expecting them to be taken seriously, does not justify my bringing it forward at this juncture.

Mr. T. A. A. Broadbent (University of Reading) : I should like to thank Mr. Dockeray for his stimulating paper, and congratulate him on retaining an optimism which I can no longer share with him. If I confess myself dubious about the success of his scheme, it is not, however, because it seems too revolutionary. The suggestion that the integral calculus should precede the differential calculus is likely to be a real stimulus ; it has obvious incidental advantages, for instance, in the possibility of treating the logarithm by its integral definition, and thereby avoiding some of the work about the logarithm and the exponential which was once fashionable but is no longer necessary. Moreover, the reversal is not without good historical warrant ; problems of areas and volumes were discussed by the Greeks, and were in the forefront of the seventeenth-century

progress to the calculus. But I am not quite sure of the position of differential equations in this scheme; even the simplest types seem to be neglected either completely or until the very end of the second year, though it is probable that a knowledge of these would be required by the pupil in applications of his mathematics to dynamics and physics.

I should like to support strongly Mr. Dockeray's plea for a Report of the Association on the teaching of analysis. The subject as a whole is perhaps not quite so flourishing in this country as it is in France, Germany, Poland or the United States. Some of the blame may lie on the teaching of the subject, and a Report would go far to remedy this state of affairs as far as the schools are concerned. Now that the Association has dealt thoroughly with arithmetic, algebra, geometry and mechanics, it would seem possible to tackle a Report on analysis; or at least on calculus, with another on analytical trigonometry.

Professor J. E. Littlewood, in writing of Ramanujan (*Gazette*, XIV, p. 426, April 1929), says: " He was not interested in rigour, which for that matter is not of first-rate importance in analysis beyond the undergraduate state, and can be supplied, given a real idea, by any competent professional ". May there not be a warning here for those who teach analysis at the school or undergraduate stage? Necessary as rigour of demonstration is, too much stress on this may stunt the development of ideas and so thwart the real purpose of the teaching of analysis. Ideas matter more than epsilons.

As we seem further than ever away from being on sure ground in dealing with the very foundations of mathematics, a pupil going up to a university might find that his knowledge of Dedekind sections was not the absolute truth he had supposed, nor am I convinced that it is really desirable to teach this subject at the school stage, even if it were possible. But that is perhaps not an adverse criticism of Mr. Dockeray's attractive if unattainable ideal.

Mr. C. V. Durell (Winchester): Mr. Dockeray's proposal that the Mathematical Association should draw up one or more Reports on the teaching of scholarship mathematics in schools will command general support. The need for an enquiry of this kind is becoming greater each year, not only because the demands made on scholarship candidates are steadily becoming more severe, but because mathematical research is exercising a definite influence on special work at schools.

Mr. Dockeray's actual proposals are ambitious, and, if I understand correctly his point of view, I cannot agree with him in thinking that university teachers are justified in complaining (*if* in fact they do so) that pupils come to them without having received any or adequate training in the fundamentals of the subject. I hold strongly that the philosophical basis of mathematics lies outside the province of the schoolmaster, even if he is qualified to deal with it; that such work is solely a matter for the teacher at the university. There must be very few schools where there is more than one pupil

per year who has the type and maturity of mind which can appreciate the Russell concept of number or the implications of the Heine-Borel theorem: in a great many schools there will not be more than one such pupil in four or five years. VIth form and scholarship work at school must be organised to suit the general requirements of pupils who need their more advanced mathematics for a variety of reasons, for physics, for engineering, for actuarial work, etc.; the pure mathematician (that is, the mathematician who will specialise in analysis or geometry at the university) is rare. The object of the schoolmaster must be to provide a general and comprehensive equipment, embracing all the aspects of pure and applied mathematics which are suitable at the school stage, with a sufficiency of technical facility, so that the pupil when he goes to the university is in a position to specialise in whatever branch of the work he desires. Obviously he must not be taught anything he has afterwards to unlearn, but avowedly conscious appeals to intuition have as much place in VIth form as in lower school work.

The methods of dealing with scholarship work at school are changing, and the more extensive use of calculus has enabled the work to be rearranged: some topics have been curtailed or dropped altogether, and new lines of attack have been developed more in keeping with the ideas of modern analysis. A report on these matters would be of great value to the schoolmaster, and might also exercise a beneficial influence on the form and content of scholarship examination papers.

Mr. F. J. Swan (Hackney Downs): Many who listened to Prof. Neville's stirring challenge in January and now have heard Mr. Dockeray's somewhat revolutionary proposals for a course in Analysis must have been forced, as I have, to re-examine our position with regard to recent advances in mathematics. The charge levelled against teachers of the more advanced mathematics in secondary schools is that their work is not up-to-date and that instead of opening up avenues they are leading their pupils into what have proved to be blind-alleys. We admit the crime but plead extenuating circumstances.

These fresh demands for reform are, under present circumstances, frankly impossible in the average secondary school. In many such schools the Senior Mathematical Master is the only out-and-out specialist on the staff. His duties include the organisation of the teaching of mathematics to pupils of ages varying from eleven (in some cases eight) to eighteen years, and the bulk of the certificate and higher certificate teaching in both pure and applied mathematics falls to his lot. The majority of the higher certificate candidates are not mathematical specialists, and the total time allowance for mathematics is in the neighbourhood of six to nine hours a week. In this limited time the prescribed syllabuses must be covered before anything else is attempted. If there happens to be a particularly good pupil, the master concerned may, if he feels disposed, give up some of his very few free periods to help such an

one. Rarely, if ever, is provision made for scholarship work or for a third-year course in such schools as those which I have in mind.

Nor do the existing syllabuses or examination papers help in this matter of reform. If our teaching is " old-fashioned ", then the examinations for which we have to prepare our pupils might well be called " archaic ". The universities are not living up to their own ideals, and we would remind them that example is better than precept.

Again, the master in the ordinary secondary school is not in touch with modern developments to the same extent as the university professor or lecturer. Those working in the universities move amongst men engaged in research and frequent interchange of ideas is possible. The secondary schoolmaster has to plough a lone furrow, for rarely is he able to meet those with whom he can discuss his problems. Here again the universities are not helpful. Even in London where one would imagine something might easily be done to help the specialist in the school to keep in touch with modern developments, the only fare offered in the evening is the post-graduate course for the M.A. degree.

It has been suggested that it is the business of the secondary school to examine carefully the logical basis of the mathematics taught. I say most emphatically that it is not our business nor are we equipped to undertake this task. We can give the ideas or, to misquote Prof. Neville, we can bring the materials to the site and see that the materials are sound, but we are not to be the builders. In conclusion, may I say that when the universities begin to pipe a new tune they will not find secondary school teachers slow to teach the new steps.

K. S. Snell (Harrow): It is useful to compare the teaching of analysis with that of elementary geometry. The latter is usually divided into three stages: A, experimental and intuitive; B, deductive and laying emphasis on groups of theorems; C, binding all the results obtained into a logical sequence and examining the axioms on which the whole is based. In analysis the Stage A work is the first-year course laid down by Mr. Dockeray, with which I am in agreement. Stage B work takes sections from the complete course and deals with them logically, stating clearly what assumptions are made. For example, assuming Rolle's theorem, the mean value theorem and Taylor's theorem can be proved, and from this various expansions can be obtained. Stage C work is the complete logical course taken in order, such as Mr. Dockeray suggests. This last stage is in my opinion beyond the scope of schools and is university work. It is for schools to lay the foundation by doing the Stage B work, but in so doing they must emphasise far more than at present that they are showing part of a complete course. This is made much easier if in the last two years of school boys can work from a book which gives a complete course. Such a book is Mr. Dockeray's, and by making judicious selection I have found it both stimulating and workable.

Mr. H. Pfannmuller (Brentwood) : I should be the last to suggest that we should teach a boy anything which he will have to unlearn later, but I cannot help feeling that there is a tendency growing up to thrust upon the schools work that properly belongs to the universities. It seems to me that real rigour in analysis belongs definitely to the university and not to the school.

I feel that what is needed is not so much excessive rigour at the stage when the boy is just beginning the subject, as a clearer realisation on the part of university teachers that their function is to teach. In my own experience the gentlemen whose lectures I attended had, with one or two honourable exceptions, no idea of lecturing.

The suggestion has been made that a committee of the Association should report on the teaching of analysis in schools. I should welcome such a report ; but if such a committee is set up, I should like its members to bear in mind all the time that they are proposing to teach boys and not undergraduates.

Mr. J. Katz (Selhurst) : There is room for at least three reports : (i) on the syllabus for the Higher School Certificate ; (ii) on a course for mathematical specialists ; (iii) on the teaching of calculus in schools.

Thirty years ago the calculus was not usually taught at school : it ranked as a university subject. Since then a teaching technique has been invented which makes it possible to introduce the calculus to a form IV or form V. Every boy, before he leaves school or before he starts to specialise in classics or modern studies, ought to know something of the calculus as a part of general intellectual culture. But would such an introductory course be possible or culturally valuable if the teacher started with the theory of sequences and put integration before differentiation ? This way of starting has, no doubt, the logical advantage of making analysis independent of geometrical intuition. But, if I draw a secant to the curve $y=x^2$ and then change the secant into a tangent, I can give a visible meaning to every step in the differentiation of x^2. Can we afford to sacrifice this psychological advantage to a demand for logical completeness ?

The pressure that is now being put upon the schoolmaster to widen the basis of his mathematical teaching is a recognition of the fact that in the school the boy is taught and not lectured, whereas, at the university, the undergraduate is too often a victim of incompetent lecturing. But, after all, the school is not the university. And there are definite limits to what the school could do by way of anticipating the specialist work of the university.

GLEANINGS FAR AND NEAR.

1029. . . . but what the mathematician predicts today has a habit of becoming what the physicist finds tomorrow.—Leader in *The Times*, October 4, 1934. [Per Miss M. J. Griffith.]

LA NOTION DE DIFFÉRENTIELLE DANS L'ENSEIGNEMENT.*

PAR J. HADAMARD.

POINCARÉ, dans sa conférence prononcée au Musée pédagogique de Paris en 1904, déclarait déjà qu'il y avait lieu de penser en dérivées et non en différentielles. Il me semble utile pour l'enseignement de se conformer résolument à ce principe et d'abandonner les explications assez compliquées qui sont classiquement données sur le symbole *d*. Pour la différentielle première, passe encore : Je puis comprendre l'égalité

(1) $$dy=f'(x)\,dx$$

ou

(1′) $$dz=p\,dx+q\,dy$$

en relation avec l'égalité approchée

(2) $$\Delta y=f'(x)\,\Delta x$$

ou

(2′) $$\Delta z=p\Delta x+q\Delta y$$

dans laquelle Δx, Δy, Δz sont des accroissements infiniment petits. Mais la différentielle seconde ! J'ai lu, comme tout le monde, l'histoire de la différentielle de la variable indépendante qui doit être constante (et qui est d'ailleurs forcément variable puisque infiniment petite). Si je me suis décidé à ne pas exposer ces considérations dans les cours que j'ai eu l'occasion de professer sur les débuts du Calcul différentiel, c'est que j'avoue ne les avoir qu'à moitié comprises moi-même.

J'entends bien qu'elles doivent être tout de même compréhensibles et que, si elles m'avaient été nécessaires pour aborder, par exemple, les applications géometriques du calcul différentiel, je serais, je l'espère, arrivé à m'en rendre maître. Mais, précisément, tel n'a jamais été le cas. J'ai étudié comme tout le monde la Géometrie infinitesimale, sans que le fait de différentier deux fois me parût introduire des difficultés spéciales et sans penser jamais, je l'affirme, à laisser constante la différentielle de la variable indépendante.

Que signifie donc l'égalité (1) ?

Tout simplement, que, x (et par conséquent $y=f(x)$) étant fonctions d'une variable auxiliaire quelconque u, on a, quelle que soit la relation entre x et u (pourvu que x'_u existe),

(3) $$\frac{dy}{du}=f'(x)\frac{dx}{du},$$

ce qui est le théorème des fonctions de fonctions.

Que signifie l'égalité (1′) ?

* Reprinted from *Scripta Universitatis atque Bibliothecae Hierosolymitanarum* (1923).

Que si x, y et, dèslors, $z=f(x, y)$ sont exprimés en fonctions d'une variable auxiliaire quelconque u, on a, quelles que soient ces expressions,

$$\frac{dz}{du}=\frac{\partial z}{\partial x}\frac{dx}{du}+\frac{\partial z}{\partial y}\frac{dy}{du}=p\frac{dx}{du}+q\frac{dy}{du}. \tag{3'}$$

Tel est le sens unique des égalités (1), (1′). Les égalités (3) et (3′) ayant lieu quelle que soit la variable indépendante u en fonction de laquelle les autres variables sont exprimées, on supprime la mention de u. L'avantage précieux de la notation différentielle consiste précisément en la possibilité de ne pas préciser quelle est la variable que l'on considérera comme indépendante.

Que signifiera l'égalité

$$d^2y=f'(x)\,d^2x+f''(x)\,dx^2 \tag{4}$$

ou

$$d^2z=pd^2x+qd^2y+rdx^2+2s\,dx\,dy+tdy^2\ ? \tag{4'}$$

Uniquement que l'on a

$$\frac{d^2y}{du^2}=f'(x)\frac{d^2x}{du^2}+f''(x)\left(\frac{dx}{du}\right)^2 \tag{5}$$

ou

$$\frac{d^2z}{du^2}=p\frac{d^2x}{du^2}+q\frac{d^2y}{du^2}+r\left(\frac{dx}{du}\right)^2+2s\frac{dx}{du}\frac{dy}{du}+t\left(\frac{dy}{du}\right)^2 \tag{5'}$$

de quelque manière que les variables aient été exprimées en fonction du paramètre u, pourvu que l'on ne cesse pas d'avoir $y=f(x)$ ou $z=f(x, y)$.

Enfin, que signifie ou que représente l'égalité

$$d^2z=rdx^2+2s\,dxdy+tdy^2\ ? \tag{6}$$

A mon avis, rien du tout. On peut, si l'on veut, noter que les deux premiers termes disparaissent au second membre de (5′) lorsque x et y sont fonctions *linéaires* de u, comme il arrive dans les démonstrations du théorème de Taylor [seule question, à ma connaissance, qui nécessite l'intervention de l'expression (6)]. En dehors de ce cas, je ne vois pas ce qu'on peut tirer de (6), . . . si ce n'est peut-être une ou deux idées fausses. J. H.

1030. Double Acrostic.

The great twin brethren. Woe that we must sing
Of one " The king is dead ", the while we bring
To one the loyal cry " Long live the king ! "

Reflect and then name — How they built up their fame.
Triangular centres — As hailed by most mentors.
In practice a guess, — And in theory—yes ?
If a converse is hard — Does it get through your guard ?
A point that is right — In American sight.
Just midway between — Two points that you've seen.
Geometrical home — Of a palindrome.

ENGLISH LETTERS AND THE ROYAL SOCIETY IN THE SEVENTEENTH CENTURY.*

By F. P. Wilson.

The Royal Society received its first Charter in 1662 from that " Defender of the Faith " and " universal lover and patron of every kind of truth ", Charles II. The movement in human thought which made its foundation inevitable had already a long history behind it. In literature the Renaissance manifested itself in a new interest in the classics of Greece and Rome and in the application of the wisdom of the ancients to modern life ; in philosophy it rejected scholasticism and declared its faith in the exercise of reason and free enquiry ; in art it delighted in the reproduction of human and natural forms ; and in science it conceived that the boundaries of man's knowledge of the universe were infinitely extendible by properly conducted experiments to the glory of God and especially to the relief of man's estate. Fundamentally the movement meant a shifting of values from the next world to this, the replacement of the doctrine of original sin by a belief in the nobility and even the perfectibility of man,† and a faith in the value of " progress ", that is, in the beneficence of the extension of man's knowledge of the natural world and of his increased control over it.

It is impossible to-day either to be as optimistic as were the early men of science in the value of their attitude and of the discoveries which sprang from it or to be as contemptuous of the scholastic tradition. It might be maintained that the twelfth and thirteenth centuries—with their cathedrals and schools and new universities, with the birth of modern European poetry in Provence, with St. Thomas Aquinas and Dante—saw a more splendid renaissance of the human spirit than any the world has witnessed since. But it is not for me at this time and in this place to join in the battle between the humanists and the mediaevalists now waging on many fronts, but rather to point to some of the characteristics and consequences of this new movement, especially in so far as it encouraged the spirit of scientific enquiry and experiment.

The first academies of literature and philosophy were founded in the wealthy cities of Northern Italy. Leonardo da Vinci (1452-1519), the greatest exemplar of a man of the Renaissance, one who took all knowledge for his province, was himself an experimenter of remarkable genius, but he formed no school or society, and the view that he established an academy of arts at Milan in which the study of geometry was much cultivated has been shown to be false. The first Society for the investigation of physical science was established at Naples in 1560 with the name *Academia Secretorum Naturæ*.‡ Italy in the sixteenth and early seventeenth centuries

* A paper read to the Yorkshire Branch of the Mathematical Association on the 17th November, 1934.

† Cf. T. E. Hulme, *Speculations* (1924), 47.

‡ C. R. Weld, *History of the Royal Society* (1848), i. 4-10.

was the most active centre of the new movement of experimental enquiry. There Copernicus, of Polish and German ancestry, spent six years and became a pupil of Novara, Professor of Mathematics and Astronomy at Bologna, and most of the men of science who increased the sum of human knowledge from Copernicus to Galileo were either Italians or had received a part of their education in that country.

To England the Renaissance, delayed by the civil struggles of the fifteenth century, came rather late than soon. When it came, the first generation of English humanists—Grocyn, More, Colet and the rest—were hardly touched by the spirit of scientific enquiry, but were drawn to the study of Greek because it was a key to treasures of literature and philosophy and religion. Sixteenth-century England, however, was not without its experimentalists, though they did not as yet form themselves into societies.* It will be sufficient to mention the names of Thomas Digges, William Gilbert, and William Harvey. Gilbert published his researches on magnetism in 1600, and started the word " electric " on its modern career. Harvey's book on the circulation of the blood appeared in 1628, " the result of many years' observation on men and living animals ".† Still earlier, in 1576, Thomas Digges, an English mathematician, ardently proclaimed his belief in the Copernican theory, and went further than Copernicus in maintaining that " the universe is infinite, that the stars are numberless, that they are located at varying distances from the centre, the sun, and extend throughout infinite space ". Nor were his theories the mere extravagances of mystical speculation, like those of Giordano Bruno a few years later. Digges's theories were controlled, as far as might be, by observation and experiment.‡

On the Continent, therefore, and in this country, there were many men of science before Bacon who were not content to accept the traditional knowledge of the ancients and who realised the necessity of experiment and observation. Bacon's stature as a man of science has shrunk with the increase of our knowledge of the history of science. Not only did he himself make no contribution of value to scientific discovery, but he ignored or refused to accept many of the most important discoveries of his time. It is clear that he was by no means acquainted with the best that was being thought in the scientific world of his day. Moreover, his Novum Organum or new instrument of knowledge appears to have inspired no one man

* The Royal College of Physicians, established in 1518 through the efforts of Thomas Linacre, is an apparent exception, but this was not primarily a society for the pursuit of knowledge for its own sake but a professional body designed to discourage quacksalvers and to reform the practices of the profession. Cf. C. R. Weld, *op. cit.* i. 73-4.

† W. C. D. Dampier-Whetham, *A History of Science* (1929), 130.

‡ The importance of Digges's work in the history of science was first pointed out by two American scholars, F. R. Johnson and S. V. Larkey, in *The Huntington Library Bulletin*, Number 5, April 1934.

of science with the possible exception of Robert Boyle.* But if his influence was almost negligible on those who were fruitfully engaged in experimental science, he did important work in popularising the gospel according to Experiment. He wrote for the most part in Latin, and his voice was heard throughout Europe. After he had spoken the new philosophy was no longer the way of life followed by a few scattered investigators: it became known to many who were no men of science, and the progress of the new movement was accelerated.

In his *Essays* where he speaks of the common relationships between man and man Bacon is cold and passionless. But when he speaks of science and its glories his speech is eloquent. In the *Essays* he is the spectator: in the philosophical works the enthusiast. The effect of his writings was greatly to enhance the dignity and prestige of science. In his most majestic paragraphs † he catches, in Wordsworth's fine phrase, "the impassioned expression which is in the countenance of all Science".

Bacon also wrote in English a popular account of the great blessings which science might confer upon mankind; and his hopes for science and the new method in science are expressed in *The New Atlantis* in his description of Solomon's House, a foundation of fellows whose end was the knowledge of causes and secret motions of things. The analogy with the Royal Society is obvious, and when Joseph Glanvill—the "grave Glanvil" of Matthew Arnold's *Scholar-Gipsy*—dedicated his *Scepsis Scientifica* to the Society in 1665 (he had been elected a fellow in the previous year), he told them: "You really are what former Ages could contrive but in wish and Romances; and Solomon's House in the *New Atlantis* was a Prophetick Scheam of the Royal Society". Solomon's House like the Royal Society was a corporate body of fellows working together for the advancement of science; it kept in touch with the work of men of science in foreign countries; it believed in things, not words—"Nullius in Verba" was and is the motto of the Royal Society; and it held that experimental science would enlarge "the bounds of Human Empire to the effecting of all things possible". The fellows of Solomon's House, says Bacon, had some degrees of flying in the air and ships and boats for going under water. They had means to convey sounds in trunks and pipes in strange lines and distances. They could make one tree or plant turn into another, and even had means of growing plants without seeds. The Royal Society had hopes which must have seemed as visionary to unsympathetic observers, and this passage from *Scepsis Scientifica* well illustrates the intellectual ferment of the time:

> Me thinks this Age seems resolved to bequeath *posterity* somewhat to remember it: The glorious Undertakers, where-

* W. C. D. Dampier-Whetham, *op. cit.* 138.

† As, for example, in the last paragraph but one of *The Advancement of Learning*, Book I, or in the fragmentary *Filum Labyrinthi*, to which Shelley refers in *A Defence of Poetry* when he calls Bacon a poet.

with Heaven hath blest our dayes, will leave the world better provided then they found it. And whereas in former times such generous free-spirited Worthies were as the Rare newly observed *Stars*, a single one the wonder of an Age: In ours they are like the lights of the greater size that twinkle in the *Starry Firmament*: And this last Century can glory in numerous *constellations*. Should those *Heroes* go on, as they have happily begun, they'll fill the world with *wonders*. And I doubt not but posterity will find many things, that are now but *Rumors*, verified into *practical Realities*. It may be some Ages hence, a voyage to the *Southern* unknown *Tracts*, yea possibly the *Moon*, will not be more strange then one to *America*. To them, that come after us, it may be as ordinary to buy a *pair* of *wings* to fly into remotest *Regions*; as now a *pair* of *Boots* to ride a *Journey*. And to conferr at the distance of the *Indies* by *Sympathetick* conveyances, may be as usual to future times, as to us in a *litterary* correspondence. The *restauration* of gray hairs to *Juvenility*, and renewing the exhausted marrow, may at length be effected without a *miracle*: And the turning of the now comparative *desert* world into a *Paradise*, may not improbably be expected from late *Agriculture*.

It cannot in fairness be maintained that Glanvill subordinated absolute values to experimental discovery, but after the experience of three centuries we may find his optimistic faith in scientific research as a contribution to the sum of human happiness a little naive. Perhaps not many of Cowley's contemporaries were brought up sharp by his statement in *A Proposition for the Advancement of Experimental Philosophy* (1661) that the three discoveries of modern times which had proved of special benefit to men were printing, guns, and America.

The original Fellows of the Society regarded Bacon as the most important of those who had prepared the way. Prefixed to Sprat's *History of the Royal Society* (1667) * is an ode to the Society by Abraham Cowley in which Bacon is saluted as the Moses who had led them forth out of the barren land of scholastic philosophy.

Bacon, like *Moses*, led us forth at last,
The barren Wilderness he past,
Did on the very Border stand
Of the blest promis'd Land,
And from the Mountain's Top of his exalted Wit,
Saw it himself, and shew'd us it.

The group of men of science from which the Royal Society immediately sprang began to meet together in London and in Oxford in the sixteen-forties. " Their first Purpose ", says Sprat, " was no more than only the Satisfaction of breathing a freer Air, and of conversing in Quiet one with another, without being ingag'd in the Passions and Madness of that dismal Age ". Among the group were

* I quote from the fourth edition (1734).

such illustrious men as Robert Boyle, John Evelyn, Robert Hooke, Christopher Wren, and William Petty. When the dismal age passed, and Charles II returned to the throne of his father, the time was ripe for the establishment of a formal Society, and at a memorable meeting held in November 1660 after the lecture was ended and those who attended "did, according to the usual manner, withdrawe for mutuall converse", it was resolved to found a college "for the promoting of Physico-Mathematicall Experimentall Learning".* The king's consent was obtained, and the Society was incorporated by Royal Charter in 1662 with the aim of advancing "the knowledge of natural things, and all useful arts by experiments".

Only a small proportion of the early members of the Society were men of science. The universities were on the whole hostile to the new philosophy, and most of the original members were gentlemen of leisure and men of affairs. But the future of a Society which included among its original fellows Robert Boyle, Robert Hooke and Christopher Wren, and within ten years of its foundation Isaac Newton, could not long be in doubt.

But the Society had its enemies and its satirists. The most serious charge which Sprat in his *History* had to rebut was that of irreligion. Sprat, who afterwards became Bishop of Rochester, made a sharp distinction, as Bacon had done before him, between the study of God's word which is divinity and the study of God's works which is philosophy, but not all fellows of the Society were content to draw a line beyond which sceptical enquiry might not proceed. Sir William Petty, one of the founders of the modern science of political economy, told Aubrey that the Society should not have chosen St. Andrew for its patron saint, but rather St. Thomas, for he would not believe till he had seen.

There was danger too from the wits and satirists of the age, and never before nor since has England been so full of men who possessed the art of making men and institutions ridiculous. The wits did not lack material. The first volume of the Society's register book contains such entries as these:

> The Duke of Buckingham † promised to bring into the Society a piece of a unicorne's horn.
>
> Sir Kenelme Digby related that the calcined powder of toades reverberated, applyed in bagges upon the stomach of a pestiferate body, cures it by severall applications.
>
> Dr. Ent, Dr. Clarke, Dr. Goddard, and Dr. Whistler, were appointed curators of the proposition made by Sir G. Talbot, to torment a man presently with the sympatheticall powder.
>
> A circle was made with powder of unicorne's horn, and a

* Weld, *op. cit.* i. 65 (cited from the first Journal Book of the Society).

† Of whom Dryden wrote:

Stiff in opinions, always in the wrong,
Was everything by starts and nothing long;
But in the course of one revolving moon
Was chymist, fiddler, statesman, and buffoon.

spider set in the middle of it, but it immediately ran out severall times repeated. The spider once made some stay upon the powder.

Sir John Finch's piece of an incombustible hat-band was produced.

The amanuensis produced artificial serpents, which being fired, and cast into the water, burnt there till they bounced.*

The wits of the age also found subjects for raillery in the Society's museum of natural curiosities, of which a catalogue was published in 1681. The catalogue is extremely curious, although the Society possessed nothing so surprising as two items in the Museum of the Tradescants, " a natural dragon " and " two feathers of the phoenix tayle ". †

In this transitional age the incongruity between the scepticism and credulity of one and the same Fellow is sometimes remarkable. In the list of original Fellows of the Society we find the name of John Aubrey, a man of insatiable curiosity in men, manners, and antiquities. Quotations from his *Brief Lives* have brightened many a dull page and " refocillated " the wasted spirits. As a biographer he had an eye for bright significant detail and a gift for sharp and singular phrasing, and as an antiquary he was remarkable for his zeal and for the accuracy of his observation. But excellent as he was as a biographer and as an antiquary, Aubrey was not distinguished as a man of science. This Fellow of the Royal Society appears to have believed in the efficacy as a cure for the ague of abracadabra if written triangularly on parchment and worn round the neck.‡

Among the earliest satirists of the Society was Thomas Shadwell in his comedy *The Virtuoso* (1676). The word " virtuoso ", recently introduced into the language, was already acquiring a depreciatory sense. The virtuoso in science corresponded to the pedant in scholarship, with the difference that he affected things, not words.§ Sir Nicholas Gimcrack is the Virtuoso in the play. " 'Tis below a Virtuoso ", says he, " to trouble himself with Men and Manners. I study Insects ".||

* T. Birch, *The History of the Royal Society of London* (1756), i. 41 and 66, and C. R. Weld, *op. cit.* i. 110-113. The experiment with unicorn's horn had been performed years earlier by William Davenant with the same result. It was believed that if the powder was from the horn of a genuine unicorn the spider would be charmed to remain within the circle. The difficulty was to secure genuine horn. Cf. John Aubrey, *Brief Lives* (ed. A. Clark), i. 205.

† C. R. Weld, *op. cit.* i. 187 and 278. In 1682 the Museum of the Tradescants became the property of the University of Oxford.

‡ *Miscellanies* (1696), 106. The Spell is " much approv'd ", and with it " one of *Wells* hath Cur'd above an Hundred of the Ague ".

§ Cf. Samuel Butler, *The Genuine Remains* (1759), ii. 186.

|| The editors of Shadwell do not seem to have noticed that the dramatist took some of his dialogue from Glanvill. Cf. with the passage cited above from *Scepsis Scientifica* Gimcrack's speech in Act II (*Complete Works*, ed. Montague Summers, iii. 126): " I doubt not but in a little time to improve the Art [of Flying] so far, 'twill be as common to buy a pair of Wings to fly to the World in the Moon, as to buy a pair of Wax Boots to ride into *Sussex* with."

Samuel Butler, who had scourged the Puritans in *Hudibras*, also felt that the Society " ran too much at that Time into the Virtuoso Taste, and a whimsical Fondness for surprizing and wonderful Stories in natural History ".* He describes in *The Elephant in the Moon* a meeting of the Society at which the members examine the full moon through a telescope. To their surprise they discover two armies fighting in the moon and an elephant much larger than any bred in Africa. They rejoice in the importance of their discovery and the fame it will bring their Society.

T'out-throw, and stretch, and to enlarge
Shall now no more be laid t'our Charge ;
Nor shall our ablest *Virtuosos*
Prove Arguments for Coffee-houses ; . . .
No more our making old Dogs young
Make Men suspect us still i'th' Wrong ;
Nor new-invented Chariots draw
The Boys to course us, without Law ;
Nor putting Pigs t'a Bitch to nurse,
To turn 'em into Mungrel-Curs,
Make them suspect, our Sculs are brittle,
And hold too much Wit, or too little :
Nor shall our Speculations, whether
An Elder-stick will save the Leather
Of Schoolboy's Breeches from the Rod,
Make all we do appear as odd.
This one Discovery's enough,
To take all former Scandals off.

When the elephant moves in a flash from the west side of the moon to the east, their wits are taxed to explain this prodigious phenomenon. But when through the common sense of a footboy the telescope is dismantled, it is discovered that the armies in the moon are swarms of flies and gnats in the telescope, and the elephant dwindles into a mouse.

There were other men of letters who were more in sympathy with the Society's aims. The most popular English poet in 1662 was Abraham Cowley, from whose *Ode to the Royal Society* a passage has already been cited. In a pamphlet published in 1661 he had advocated the foundation of a college for the advancement of experimental philosophy, not a visionary scheme like Bacon's, which he called a project for experiments that can never be experimented, but a scheme that only needed sufficient endowment to make it practicable. The Fellows of his college would have been required to take an oath never to write anything to the college " but what after very diligent Examination, they shall fully believe to be true, and to confess and recant it as soon as they find themselves in an Errour ". Cowley would have attached a school to his college, and Sprat's objection to this proposal should be remembered by all who are engaged in the art of teaching. It would devour too much of the Fellows' time, and " it would go near to make them a little more *magis-*

* *Op cit.* i. 1.

terial in Philosophy, than became them ; by being long accustomed to command the Opinions, and direct the Manners, of their Scholars ".

Many of the early Fellows of the Society made their mark more in the history of literature than of science. There are the diarists, John Evelyn and Samuel Pepys. Evelyn twice declined the office of President, although " much importuned ". Pepys was P.R.S. from 1684 to 1686. In March 1665, however, after attending a meeting of the Society, he confessed, in the privacy of his diary, that there were " fine discourses and experiments, but I do lack philosophy enough to understand them, and so cannot remember them ". Also members were John Aubrey and the poet Sir John Denham. It is on record that in 1682 another poet, Edmund Waller, excused himself for not paying his annual dues by pleading that he had never been able to attend the meetings of the Society, " being perpetually in parliament ".*

We should not expect to find John Milton in this company even if he had been eligible on political grounds. The system of education which he recommended in the famous letter to Master Samuel Hartlib was catholic enough to include a study of natural philosophy, but this study was preliminary and subordinate. The real end of learning " is to repair the ruins of our first parents by regaining to know God aright ". Although Milton had visited Galileo in Italy, and although the " Tuscan artist " and his " optic glass " are mentioned in *Paradise Lost*, it is significant that when Adam enquires concerning celestial motions in Book VIII of the epic, he is doubtfully answered by the affable archangel Raphael and " exhorted to search rather things more worthy of knowledge ".

With the exception of Milton, however, the four most famous poets of the day were associated with the Royal Society—Cowley,† Waller, Denham, and, greatest of all, John Dryden. Dryden took an active interest in the Society in its early days. In his verses to another original Fellow, Dr. Walter Charleton, Dryden rejoiced in the overthrow of Aristotle's authority and praised the scientific discoveries of Englishmen.

> Among th' *Assertors* of free Reason's claim,
> Th' *English* are not the least in Worth, or Fame.
> The World to *Bacon* does not only owe
> Its *present* Knowledge, but its *future* too.
> *Gilbert* shall live, till *Load-stones* cease to draw,
> Or *British* Fleets the boundless Ocean awe.
> And noble *Boyle*, not less in *Nature* seen,
> Than his great *Brother* read in *States* and *Men*.
> The *Circling* streams, once thought but pools, of blood
> (Whether Life's fewel, or the Bodie's food)
> From dark Oblivion *Harvey*'s name shall save ;
> While *Ent* keeps all the honour that he gave.

* T. Birch, *op. cit.* iv. 130.

† All except Cowley were original Fellows. Cowley was proposed for membership and elected in 1661, but the records do not show that he was ever admitted or attended any of the meetings. See A. Nethercot, *Abraham Cowley* (1931), 217.

Again, into his *Annus Mirabilis* (1667) Dryden puts an apostrophe to the Royal Society, and in lines that are better poetry than science praises their enquiries into tides and longitudes and the benefits these are likely to bring to English commerce.

> Then we upon our Globes last verge shall go,
> And view the Ocean leaning on the Sky :
> From thence our rolling Neighbours we shall know,
> And on the Lunar world securely pry.

It is the early Dryden, Dryden the " metaphysical " poet, who actively identified himself with the Royal Society. Sprat stated that the discoveries of modern science would provide the poets with a new storehouse of images, but the later " Augustan " Dryden would not be so likely to draw on this storehouse as the early Dryden. In the *Annus Mirabilis* (1667) Dryden was at pains to use the proper sea-terms in his description of a naval fight, and was not merely content, like other poets, to employ such " common actions " as "the thundering guns, the smoke, the disorder, and the slaughter " ; but in the *Dedication of the Æneis* (1697) he pointed out that Virgil did not write always " in the proper terms of navigation, land-service, or in the cant of any profession ", and the reason was that he wrote " not to mariners, soldiers, astronomers, gardeners, peasants, etc., but to all in general, and in particular to men and ladies of the first quality, who have been better bred than to be too nicely knowing in the terms ".*

Dryden is here expressing the ideal of diffused and polite learning characteristic of the next age. To the wits of Queen Anne's reign, with their love of the general and dislike of the particular, it seemed that the researches of the Fellows into the minutiæ of nature were an offence to polite society. " When I meet ", writes Steele (*Tatler* 236), " with a young fellow that is an humble admirer of the sciences, but more dull than the rest of the company, I conclude him to be a Fellow of the Royal Society ". Better known is Swift's satire on the Academy of Lagado with its projects for extracting sunbeams out of cucumbers, for calcining ice into gunpowder, and for building houses from the roof downwards to the foundation. And in Pope's *Dunciad* among the first upon whom Queen Dulness confers her degrees are those who

> Impale a Glow-worm, or Vertú profess,
> Shine in the dignity of F.R.S.

With one aspect of the Society's work, however, Dryden must

* *Essays of John Dryden*, ed. W. P. Ker, i. 13 and ii. 236. Dryden was ejected from the Society in 1666, not because he disapproved of it—the apostrophe in *Annus Mirabilis* was published later—but through failure to pay his dues. But although he does not attack the Society, in later years he seems to have lost interest in it. His conversion to Roman Catholicism may have influenced his attitude.

> Rest then, my soul, from endless anguish freed :
> Nor sciences thy guide, nor sense thy creed.
> (*The Hind and the Panther*, i. 146-7.)

have remained in sympathy till his death. In 1664 the Society, perhaps at Dryden's suggestion, appointed " a committee for improving the English tongue, and particularly for philosophical purposes ". Dryden, we know, advocated the foundation of an English Academy similar to the French Academy which had been founded in 1635, and it would have published a dictionary and a grammar for the better keeping of the English language.* The Committee of the Royal Society met a few times, then crumbled away before the Plague and the Fire. But the Society made its influence felt on the writing of English prose. This quotation from Sprat's *History* is an important document in the history of our prose style. After attacking " vicious Abundance of *Phrase* ", " Trick of *Metaphors* " and " Volubility of *Tongue* " in modern authors, Sprat proceeds :

> The Society " have therefore been more rigorous in putting in Execution the only Remedy, that can be found for this *Extravagance* ; and that has been a constant Resolution, to reject all the Amplifications, Digressions, and Swellings of Style ; to return back to the primitive Purity and Shortness, when Men deliver'd so many *Things*, almost in an equal Number of *Words*. They have exacted from all their Members, a close, naked, natural way of Speaking ; positive Expressions, clear Senses ; a native Easiness ; bringing all Things as near the mathematical Plainness as they can ; and preferring the Language of Artizans, Countrymen, and Merchants, before that of Wits, or Scholars."

In prose and in poetry English literature about the time of the Restoration underwent one of the most sudden revolutions in taste in our history. In prose the change was much more than a matter of style, but expressed in terms of style it was a revulsion from the ornate rhetoric with Latinised diction and elaborate sentence-architecture of such writers as Milton and Browne and Jeremy Taylor. The case of Sir Thomas Browne is most interesting because he has a foot in both worlds. He was himself a man of science, an enthusiast for the new world of knowledge and the " Exantlation of Truth " by joining sense to reason and experiment to speculation.† His largest work was perhaps inspired by Bacon. Bacon in *The Advancement of Learning* had noted the lack of " any due rejection of fables and popular errors ", and had recommended their examination and refutation.‡ This task Browne set himself in *Pseudodoxia Epidemica* or *Vulgar Errors* (1646). From this work we may learn whether bitter almonds are indeed a preservative against Ebriety, whether an Elephant has no joints, whether Adam

* Cf. O. F. Emerson, " John Dryden and a British Academy " (*Proceedings of the British Academy*, vol. x.).

† *Christian Morals*, ii. 5.

‡ No doubt this was what the Society was at in some of the experiments cited above.

and Eve should be painted without navels, or whether mandrakes naturally grow under gallows and places of execution and whether the root gives a shriek upon eradication.

But Browne's mind was not solely given over to reason and sceptical enquiry. It had room enough for the world of wonder and the world of reason, for fable and fact, for delight in the natural and the supernatural. " There is in these workes of nature, which seeme to puzle reason, something Divine, and hath more in it then the eye of a common spectator doth discover." Browne shared the power of some seventeenth-century poets of fusing together heterogeneous materials. Ancient learning lay down happily in his mind with modern and both enriched his sensibilities.*

In the *Vulgar Errors*, the more absurd the errors which he examines the more delight he takes in discussing them. His aim is scientific truth, but his aim is at times obscured by the cloud of learning, the parade of authorities, which he delights to marshal. He lingers at his task of demolition and kills his errors as if he loved them. He likes to cite the authority of the ancients, to hazard a wide solution, to admit a wavering conjecture. In a way he prefers experiment to authority. He dismisses the view that moles have no eyes by saying " that they have eyes in their head is manifest unto any, that wants them not in his own ". But he does not always resort to experiment when experiment was desirable and possible. For example, consider his treatment of the problem whether badgers have the legs on one side shorter than on the other. First, the authority of Aldrovandus and Albertus Magnus is against it.

> Again, It seems no easie affront unto Reason, and generally repugnant unto the course of Nature ; for if we survey the total set of Animals, we may in their legs, or Organs of progression, observe an equality of length, and parity of Numeration ; that is, not any to have an odd legg, or the supporters and movers of one side not exactly answered by the other. Although the hinder may be unequal unto the fore and middle legs, as in Frogs, Locusts, and Grashoppers ; or both unto the middle, as in some Beetles and Spiders, as is determined by Aristotle, *De incessu Animalium*. Perfect and viviparous quadrupeds, so standing in their position of proneness, that the opposite joints of Neighbour-legs consist in the same plane ; and a line descending from their Navel intersects at right angles the axis of the Earth. . . .
>
> Lastly, The Monstrosity is ill contrived, and with some disadvantage ; the shortness being affixed unto the legs of one side, which might have been more tolerably placed upon the thwart or Diagonial Movers. For the progression of quadrupeds being performed *per Diametrum*, that is the cross legs moving or resting together, so that two are always in motion, and two

* On Browne's capacity for living in both worlds see Basil Willey, *The Seventeenth Century Background* (1934), ch. iii.

in station at the same time; the brevity had been more tolerable in the cross legs. For then the Motion and station had been performed by equal legs; whereas herein they are both performed by unequal Organs, and the imperfection becomes discoverable at every hand.

Sir Thomas Browne was never a member of the Royal Society. His style would have disqualified him. To us this style, especially in such works as *Religio Medici* and *Hydriotaphia*, may seem the reflection of a rich and adorned mind: to those who valued prose as an instrument for the communication of scientific ideas it must have seemed a blunted tool. The hour for a new prose was at hand, a prose that should be plain, rational, and easily intelligible. The period of the Restoration saw the birth of modern prose, and the Royal Society was not the least important among the influences which helped to shape that prose and to foster the writing of it.

F. P. W.

1031. In these mechanical and mathematical days how long will it be before an Engineer-Admiral paces his 25% deck? If 25% Sessions are not abolished, they will soon be trying a man for murdering his better 50% while he was 50% seas over through drinking 50-50, or perhaps 200% whiskies.—Leader in *The Times*, March 28, 1935. [Per Prof. E. H. Neville.]

1032. But Dr. Wybard in his Tectometry, Page 289, doth suppose the *Wine Gallon* to contain but 224, or 225 Cubick Inches at the most, and pursuant to this Account an Experiment was made by Mr. Richard Walker and Mr. Philip Shales, two General Officers in the Excise. They caused a Vessel to be very exactly made of Brass, in Form of a Parallelopipedon, each side of it's Base was 4 Inches, and it's Depth 14 Inches; so that it's just Content was 224 Cubick Inches. This Vessel was produced at Guild-Hall, in London (May 25, 1688) before the Lord Mayor, the Commissioners of Excise, the Reverend Mr. Flamstead, Astr. Reg. Mr. Halley, and several other ingenious Gentlemen, in whose Presence Mr. Shales did exactly fill the aforesaid Brass Vessel with clear Water, and very carefully emptied it into the old Standard *Wine Gallon* kept in Guild-Hall, which did so exactly fill it that all then present were fully satisfied the Wine Gallon doth contain but 224 Cubick Inches. (*This notable Experiment I saw tried.*) However, for several Reasons, it was at that time thought convenient to continue the former supposed content of 231 Cubick Inches to the *Wine Gallon*, and that all computations in gauging should be made from thence.—John Ward, *The Young Mathematician's Guide* (12th edition, 1771; ed. pr. 1707). [Per Miss F. A. Yeldham.]

1033. To the Honourable Samuel Pepys, Esq.; Principal Officer of the Navy, Secretary to the Admiralty, and President of the Royal Society.

Honoured Sir, The Countenance and Encouragement, you have always given to Mathematical Learning, especially as it hath a tendency to Promote the Publick Good; has emboldened me, humbly to Present your Honour, with this little Peace; which hath the Admirable Euclid for its author, and the Learned D'Chales for the Commentator. . . . *The Elements of Euclid, Explained and Demonstrated in a New and More Easy Method than any yet extant. With the Uses of each Proposition in all the Parts of the Mathematicks.* By Claude Francois Milliet D'Chales, a Jesuite. Done out of French, Corrected, Augmented, and Illustrated by Nine Copper-Plates, with the Effigies of Euclid, by Reeve Williams, Philomath (4th edition, 1732 ?). [Per Prof. E. H. Neville.]

THE PILLORY.

"A BODY is moving in a straight line with constant acceleration. Prove that, in any interval of time, the change of kinetic energy is equal to the work done by the force causing the acceleration.

A train travelling at 45 M.P.H. picks up 200 c. ft. of water from a trough $\frac{1}{4}$ mile long. Find the extra horse-power that must be generated so that the speed may be maintained. Neglect the height to which the water is raised." Northern Univ. H.S.C., 1933, Applied Mathematics.

The horse-power required is independent of the height to which the water is raised. J. LISTER.

"Three equal thin uniform straight rods AB, BC and CD, each of mass M and length $2a$, are smoothly jointed at B and C. They are laid on a smooth horizontal table so that $ABCD$ is a straight line. If the rod BC is struck at its mid-point a horizontal blow P perpendicularly to BC, prove that the rods AB and CD collide after time $4\pi Ma/3P$." Mathematical Tripos, Part I (Old Regulations), 1934.

The initial angular velocity of one of the end rods is $P/2Ma$, and the result seems to have been reached by dividing the angle $2\pi/3$ by this spin. But the angular velocity is not constant in the free motion: if there were no forces at B and C, the centres of gravity of the rods AB and CD would not approach each other. There is no need to analyse the forces, for the equations of linear momentum and energy are sufficient to determine the motion. When AB has turned through an angle θ, $\omega^2(1+\sin^2\theta)$ is constant, and therefore equal to $\omega_0{}^2$, and the number to be divided by ω_0 is not the angular measure $2\pi/3$, but the elliptic integral

$$\int_0^{2\pi/3} \sqrt{(1+\sin^2\theta)}\,d\theta.$$

E. H. NEVILLE.

"A bookseller buys six dozen copies of a book and marks them at the price of 7s. 6d., which gives him a profit of 25 per cent. on his outlay.

He sells four-fifths of his stock at the price marked, and then sells the remainder at a reduction of 20 per cent. on the marked price. What percentage of profit does he make over all?" Leaving Certificate Examination of the Scottish Education Department, 1935; Lower Grade Mathematics; Qn. 1 of Section I; all questions of this section to be attempted.

Four-fifths of six dozen is $57\frac{3}{5}$. Candidates obtaining this figure would suppose they were in error, and waste time in trying to trace their mistake. Moreover, the answer can be obtained without using either 7s. 6d. or six dozen. J. W. STEWART.

THEOREMS ON THE TETRAHEDRON.

By R. T. Robinson.

1. Through any point O in space a straight line POP_1 can be drawn meeting the opposite edges AD, BC of a tetrahedron $ABCD$ in the points P, P_1 respectively ; another straight line QOQ_1 can be drawn to meet DB, AC in Q, Q_1 ; and another straight line ROR_1 to meet DC, AB in R, R_1.

If O coincides with G, the centroid of the tetrahedron, the six associated points P, P_1, Q, Q_1, R, R_1 are then the mid-points of the edges of the tetrahedron. If the opposite edges of the tetrahedron are perpendicular to each other, there are four spheres, connected with the tetrahedron, which have the same radical plane, namely: the sphere $ABCD$, the self-polar sphere of $ABCD$, the sphere through the mid-points of the edges of $ABCD$, and the sphere through the centroids of the faces of $ABCD$. In this case the coordinates of the centroid are $(1/A, 1/B, 1/C, 1/D)$ and the equation of the absolute plane is $A\alpha+B\beta+C\gamma+D\delta=0$.

2. If we take any point O in space whose coordinates with reference to $ABCD$ are $(\alpha_0, \beta_0, \gamma_0, \delta_0)$, and if we draw through O the lines POP_1, QOQ_1, ROR_1, meeting the pairs of opposite edges in P, P_1, etc., then if we take the equation of the absolute plane to be

$$\alpha/\alpha_0+\beta/\beta_0+\gamma/\gamma_0+\delta/\delta_0=0,$$

the points P, P_1, Q, Q_1, R, R_1, will be the " middle points " of the edges of the tetrahedron with reference to this absolute plane, and there will be four " spheres " forming a coaxal system, namely, four conicoids intersecting in the same two planes, one of these planes being the absolute plane.

3. Writing D_2 for O, Fig. 1 is obtained from the four points $ABCD$, the point D_2 and the six associated points P, P_1, Q, Q_1, R, R_1 in the following manner. Taking $ABCD$ as tetrahedron of reference and the coordinates of D_2 to be $(\alpha_0, \beta_0, \gamma_0, \delta_0)$,

the coordinates of P are $(\alpha_0, 0, 0, \delta_0)$, of P_1 $(0, \beta_0, \gamma_0, 0)$;
the coordinates of Q are $(0, \beta_0, 0, \delta_0)$, of Q_1 $(\alpha_0, 0, \gamma_0, 0)$;
the coordinates of R are $(0, 0, \gamma_0, \delta_0)$, of R_1 $(\alpha_0, \beta_0, 0, 0)$.

Take points A_2, B_2, C_2 such that

$$(PD_2P_1A_2)=-1, \quad (QD_2Q_1B_2)=-1, \quad (RD_2R_1C_2)=-1,$$

then the coordinates of A_2 are $(\alpha_0, -\beta_0, -\gamma_0, \delta_0)$,
the coordinates of B_2 are $(-\alpha_0, \beta_0, -\gamma_0, \delta_0)$,
the coordinates of C_2 are $(\alpha_0, \beta_0, -\gamma_0, -\delta_0)$.

Now take points π, π_1 ; τ, τ_1 ; ψ, ψ_1, such that

$$(DPA\pi)=-1, \quad (CP_1B\pi_1)=-1 ;$$
$$(DQB\tau)=-1, \quad (CQ_1A\tau_1)=-1 ;$$
$$(DRC\psi)=-1, \quad (AR_1B\psi_1)=-1.$$

The coordinates of π are $(\alpha_0, 0, 0, -\delta_0)$, of π_1 $(0, \beta_0, -\gamma_0, 0)$;
the coordinates of τ are $(0, \beta_0, 0, -\delta_0)$, of τ_1 $(\alpha_0, 0, -\gamma_0, 0)$;
the coordinates of ψ are $(0, 0, \gamma_0, -\delta_0)$, of ψ_1 $(\alpha_0, -\beta_0, 0, 0)$.

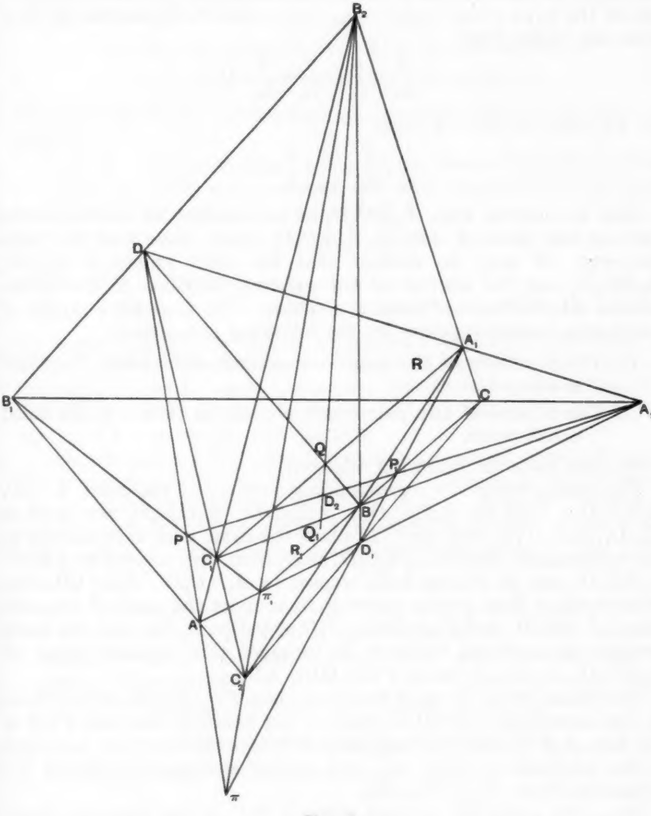


Fig. 1.

From this it follows that

π and π_1 lie on B_2C_2 and $(B_2\pi_1C_2\pi) = -1$;
τ and τ_1 lie on C_2A_2 and $(A_2\tau C_2\tau_1) = -1$;
ψ and ψ_1 lie on A_2B_2 and $(B_2\psi_1A_2\psi) = -1$.
AB_2, DC_2 intersect at C_1 $(\alpha_0, \beta_0, -\gamma_0, \delta_0)$;
DB_2, AC_2 intersect at B_1 $(\alpha_0, -\beta_0, \gamma_0, \delta_0)$;

C_2B, B_2C intersect at A_1 $(-\alpha_0, \beta_0, \gamma_0, \delta_0)$;

CC_2, BB_2 intersect at D_1 $(\alpha_0, \beta_0, \gamma_0, -\delta_0)$.

4. Thus when $ABCD$ is the tetrahedron of reference, the coordinates of the vertices of the tetrahedra $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ are of the type $(\pm\alpha_0, \pm\beta_0, \pm\gamma_0, \pm\delta_0)$, and the equations of their faces are of the type

$$\pm\frac{\alpha}{\alpha_0}\pm\frac{\beta}{\beta_0}\pm\frac{\gamma}{\gamma_0}\pm\frac{\delta}{\delta_0}=0,$$

the equation of $A_2B_2C_2$ being

$$\frac{\alpha}{\alpha_0}+\frac{\beta}{\beta_0}+\frac{\gamma}{\gamma_0}+\frac{\delta}{\delta_0}=0.$$

Had we started with $A_2B_2C_2D_2$ as tetrahedron of reference, the vertices and faces of $ABCD$, $A_1B_1C_1D_1$ would have had the same property. It may be noticed that the eight points $A_1B_1C_1D_1$, $A_2B_2C_2D_2$ are the centres of the spheres inscribed in the tetrahedron $ABCD$ if one of them is a centre. We thus get a triplet of tetrahedra bound together by the following properties:

(i) The 8 vertices of any pair lie on a conicoid to which the third is self-polar.

(ii) The 8 faces of any pair touch a conicoid to which the third is self-polar.

We also have the following relations.

The lines joining the corresponding vertices of the pairs $ABCD$, $A_1B_1C_1D_1$; $ABCD$, $A_2B_2C_2D_2$; $A_1B_1C_1D_1$, $A_2B_2C_2D_2$ intersect in D_2, D_1, and D respectively. D_1 plays the same part with respect to the tetrahedron $ABCD$, $A_2B_2C_2D_2$ as D_2 does with respect to $ABCD$, $A_1B_1C_1D_1$ and as D does with respect to $A_1B_1C_1D_1$, $A_2B_2C_2D_2$, the three straight lines drawn through D_1 to meet the pairs of opposite edges of $ABCD$, $A_2B_2C_2D_2$ being $P_1D_1\pi$, $Q_1D_1\tau$, $R_1D_1\psi$, and the three straight lines drawn through D to meet the opposite edges of $A_1B_1C_1D_1$, $A_2B_2C_2D_2$ being $PD\pi$, $QD\tau$, $RD\psi$.

The planes of the faces of the octahedron $PP_1QQ_1RR_1$ are the faces of the tetrahedra $ABCD$, $A_1B_1C_1D_1$, (for example the face PQR is the face $A_1B_1C_1$) and the diagonals of this octahedron are the edges of the tetrahedron $A_2B_2C_2D_2$, and similar statements hold for the octahedra $P_1\pi Q_1\tau R_1\psi$, $P\pi Q\tau R\psi$.

From the figure we see that when $A_2B_2C_2$ is the absolute plane, P, P_1, Q, Q_1, R, R_1 are the "mid-points" of the edges AD, BC, ...

5. Taking $ABCD$ as the tetrahedron of reference and $(\alpha_0, \beta_0, \gamma_0, \delta_0)$ as the coordinates of D_2, the equation to any conicoid through P, P_1, Q, Q_1, R, R_1 is

$$(\Sigma u\alpha\alpha_0)(\Sigma\alpha/\alpha_0)-2\Sigma u\alpha^2=0, \quad\text{........................(1)}$$

where $\Sigma u\alpha^2 \equiv u\alpha^2+v\beta^2+w\gamma^2+k\delta^2$. Incidentally this is a cone if either $\Sigma u\alpha_0^2=0$ or $\Sigma\frac{1}{u\alpha_0^2}=0$.

If $ABCD$ is a tetrahedron with its opposite edges perpendicular to each other, that is, if $a^2+d^2=b^2+e^2=c^2+f^2$, there are two positions of the point D_2 for which the conicoid (1) is a sphere.

(i) If D_2 is at G, that is, if the points $(\alpha_0, \beta_0, \gamma_0, \delta_0)$ and $(1/A, 1/B, 1/C, 1/D)$ coincide, and if $u=A^2(b^2+c^2-a^2), \dots, k=D^2(e^2+f^2-a^2)$, equation (1) becomes

$$\{\Sigma A(b^2+c^2-a^2)\alpha\}\{\Sigma A\alpha\}-2\Sigma A^2(b^2+c^2-a^2)\alpha^2=0$$

or

$$\tfrac{1}{4}\{\Sigma A(b^2+c^2-a^2)\alpha\}\{\Sigma A\alpha\}-\Sigma c^2AB\alpha\beta=0,$$

the equation of the sphere through the mid-points of the edges of $ABCD$.

(ii) If D_2 is the intersection of the perpendiculars from the vertices of $ABCD$ on to the opposite faces, that is, if the points $(\alpha_0, \beta_0, \gamma_0, \delta_0)$ and $(1/A(b^2+c^2-a^2), \dots)$ coincide and if

$$u=A^2(b^2+c^2-a^2), \dots .$$

Equation (1) becomes

$$\Sigma A\alpha \,.\, \Sigma A(b^2+c^2-a^2)\alpha-2\Sigma A^2(b^2+c^2-a^2)\alpha^2=0,$$

the same equation as before, as should evidently be the case, for in this type of tetrahedron the sphere through the mid-points of the edges is cut by each face in the nine-points circle of that face and the points $P, P_1, \dots$ are the feet of the perpendiculars from a vertex of a face to the opposite side.

There is yet another case where the associated points $P, P_1, \dots$ lie on a sphere. If the tetrahedron $ABCD$ is such that a sphere can be drawn to touch all its edges, that is, if $a+d=b+e=c+f$, the three straight lines joining the points of contact of the sphere with opposite edges pass through the same point. The points $(\alpha_0, \beta_0, \gamma_0, \delta_0)$ and

$$\left(\frac{1}{A(b+c-a)}, \dots, \frac{1}{D(e+f-a)}\right)$$

coincide, and if we take $u\equiv A^2(b+c-a)^2, \dots k=D^2(e+f-a)^2$, equation (1) becomes

$$[\Sigma A(b+c-a)\alpha]^2-2\Sigma A^2(b+c-a)^2\alpha^2=0,$$

or

$$\Sigma A\alpha \,.\, \Sigma A(b+c-a)^2\alpha-4\Sigma c^2AB\alpha\beta=0.$$

6. We can write $\Sigma u\alpha^2=0$ in the form

$$\Sigma u\alpha\alpha_0 \,.\, \Sigma\frac{\alpha}{\alpha_0}-\Sigma(v\beta_0{}^2+w\gamma_0{}^2)\frac{\beta\gamma}{\beta_0\gamma_0}=0,$$

and it is to be noted that when $u\equiv A^2(b^2+c^2-a^2), \dots$ and the points $(\alpha_0, \beta_0, \gamma_0, \delta_0)$, $(1/A, 1/B, 1/C, 1/D)$ coincide, then

$$\Sigma(v\beta_0{}^2+w\gamma_0{}^2)\frac{\beta\gamma}{\beta_0\gamma_0}=0$$

becomes

$$2\Sigma a^2BC\beta\gamma=0,$$

the equation of the sphere $ABCD$.

We take $\Sigma(v\beta_0^2+w\gamma_0^2)\dfrac{\beta\gamma}{\beta_0\gamma_0}=0$ as the equation of the "sphere" $ABCD$, for its section by the absolute plane is met by opposite edges of the tetrahedron in conjugate points. The equation of the "sphere" through the "mid-points" of the edges can be written

$$\tfrac{1}{2}\Sigma u\alpha\alpha_0 \,.\, \Sigma\frac{\alpha}{\alpha_0}-\Sigma(v\beta_0^2+w\gamma_0^2)\frac{\beta\gamma}{\beta_0\gamma_0}=0.$$

7. The straight lines joining the vertices of the tetrahedron $ABCD$ to the point $(\alpha_0, \beta_0, \gamma_0, \delta_0)$ meet the opposite faces in $(0, \beta_0, \gamma_0, \delta_0)$, $(\alpha_0, 0, \gamma_0, \delta_0)$, ... The equation to the "sphere" through these four points—that is, the conicoid corresponding to the sphere through the centroids of the faces of the tetrahedron $ABCD$—is

$$\tfrac{2}{3}\,\Sigma u\alpha\alpha_0 \,.\, \Sigma\frac{\alpha}{\alpha_0}-\Sigma(v\beta_0^2+w\gamma_0^2)\frac{\beta\gamma}{\beta_0\gamma_0}=0.$$

8. These four "spheres" form a coaxal system, the equation of the "radical" plane being $\Sigma u\alpha\alpha_0=0$, and their centres—that is, the poles of $\Sigma\dfrac{\alpha}{\alpha_0}=0$—are collinear and form a harmonic range. The "centre" of the self-polar "sphere" is $(1/u\alpha_0, 1/v\beta_0, 1/w\gamma_0, 1/k\delta_0)$. The "centre" of the "sphere" through the "mid-points" of the edges is $(\alpha_0, \beta_0, \gamma_0, \delta_0)$.

The "centre" of the "sphere" $ABCD$ is

$$\left(\frac{1}{\Sigma u\alpha_0^2}+\frac{2}{2u\alpha_0^2}\right)\alpha_0, \dots .$$

The "centre" of the "sphere" corresponding to the sphere through the centroids of the faces is

$$\left(\frac{1}{\Sigma u\alpha_0^2}-\frac{2}{u\alpha_0^2}\right)\alpha_0, \dots .$$

The equation of the polar of the centre of the "sphere" through the "mid-points" of the edges with respect to the self-polar "sphere" is $\Sigma u\alpha\alpha_0=0$, and this is the equation of the radical plane of the "spheres". This is what we should expect, as the sphere through the mid-points of the edges of the tetrahedron $ABCD$ cuts the self-polar sphere of $ABCD$ orthogonally, as can be proved geometrically by the method adopted in § 10 below.

9. If A, B, C, D and A_1, B_1, C_1, D_1 lie on a sphere, then by substituting the values of the coordinates of A_1, B_1, C_1, D_1 in the equation $\Sigma c^2AB\alpha\beta=0$ we get

$$c^2AB\alpha_0\beta_0=f^2CD\gamma_0\delta_0\,;$$
$$a^2BC\beta_0\gamma_0=d^2AD\alpha_0\delta_0\,;$$
$$b^2AC\gamma_0\alpha_0=e^2BD\beta_0\delta_0.$$

From these we get

$$\pm\frac{A\alpha_0}{aef}=\pm\frac{B\beta_0}{bdf}=\pm\frac{C\gamma_0}{cde}=\pm\frac{D\delta_0}{abc}$$

or

$$\pm\alpha_0/R_1=\pm\beta_0/R_2=\pm\gamma_0 R_3=\pm\delta_0/R_4$$

where R_1, R_2, R_3, R_4 are the radii of the sections of the sphere by the faces BCD, CDA, DAB, ABC of $ABCD$.

Thus $(\pm R_1, \pm R_2, \pm R_3, \pm R_4)$ are the coordinates of A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2 with respect to the tetrahedron $ABCD$.

In the same way the coordinates of A, B, C, D, A_2, B_2, C_2, D_2, with respect to the tetrahedron $A_1B_1C_1D_1$ are $(\pm R_1', \pm R_2', \pm R_3', \pm R_4')$, where R_1' is the radius of the section of the sphere $ABCD\,A_1B_1C_1D_1$ by the face $B_1C_1D_1$ of the tetrahedron $A_1B_1C_1D_1$, etc.

10. In the cuboidal solid $PQRSP_1Q_1R_1S_1$ (Fig. 2) the diagonal lines of any two opposite faces $PQRS$, $P_1Q_1R_1S_1$ form a tetrahedron

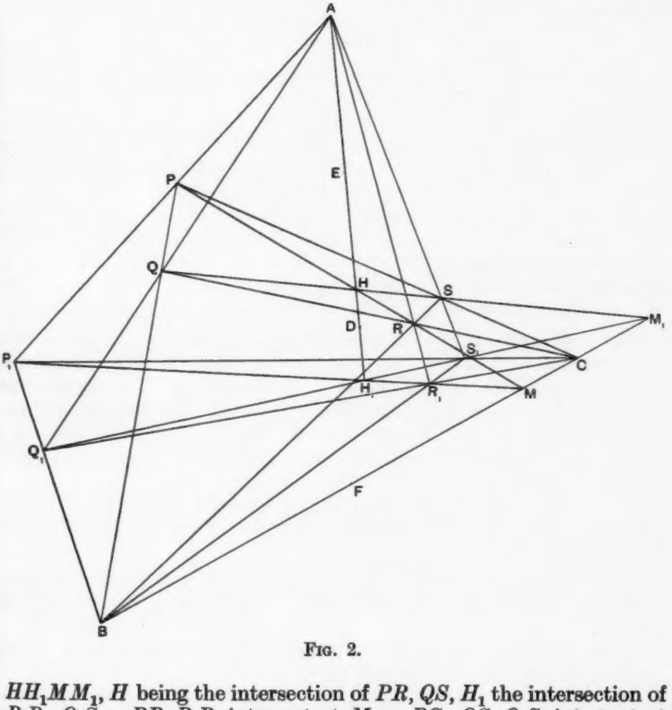


FIG. 2.

HH_1MM_1, H being the intersection of PR, QS, H_1 the intersection of P_1R_1, Q_1S_1; PR, P_1R_1 intersect at M on BC, QS, Q_1S_1 intersect at M_1 on BC, where $(BMCM_1)=-1$ and $(AHDH_1)=-1$. We prove

that, if $ABCD$ has its opposite edges perpendicular, the sphere through the mid-points of the edges of $ABCD$ cuts the sphere HH_1MM_1 orthogonally.

Preliminary. If two sphere centres C and D cut orthogonally and if AB is any diameter of the sphere centre C and APB is one of the two tangents planes through ACB touching the sphere centre D at P, then P lies on both spheres : APB is a right angle and

$$AP^2+PB^2=AB^2.$$

Conversely, if the sum of the squares of the tangents to a sphere centre D from the extremities of a diameter AB of a sphere centre C be equal to AB^2, the spheres cut orthogonally.

Let E, F be the mid-points of AD, BC respectively.

$$(AHDH_1)=-1.$$

Thus $ED^2=EH\,.\,EH_1$, and so the square of the tangent from E to the sphere $HH_1MM_1=\frac{1}{4}d^2$.

$$(BMCM_1)=-1.$$

Thus $FC^2=FM\,.\,FM_1$, and the square of the tangent from F to the sphere $HH_1MM_1=\frac{1}{4}a^2$.

But $EF^2=\frac{1}{4}(a^2+d^2)$. Thus the sum of the squares of the tangents from E and F to the sphere $HH_1MM_1=EF^2$. Thus the sphere on diameter EF—that is, the sphere through the mid-points of the edges of $ABCD$—cuts the sphere HH_1MM_1 orthogonally.

Similarly the sphere through the mid-points of the edges of $ABCD$ cuts orthogonally the sphere circumscribing the tetrahedron formed by the intersections of the diagonals of the faces PP_1, QQ_1 ; RR_1, SS_1 and the sphere circumscribing the tetrahedron formed by the intersections of the diagonals of the faces PP_1, SS_1 ; QQ_1, RR_1.

11. Turning to Fig. 1, we see that the tetrahedron $PP_1\pi\pi_1$ is the tetrahedron formed by the diagonals of the two opposite faces ADB_1C_1, A_1D_1BC. Using the same method of proof as in the last section, we see that if the opposite edges of the tetrahedron $PP_1\pi\pi_1$ are perpendicular, the sphere through the mid-points of the edges of the tetrahedron $PP_1\pi\pi_1$ cuts orthogonally the spheres $ABCD$, $A_1B_1C_1D_1$, $A_2B_2C_2D_2$.

From the figure we see that if the opposite edges of the tetrahedron $PP_1\pi\pi_1$ are perpendicular, AD must be perpendicular to BC, A_1D_1 to B_1C_1, and A_2D_2 to B_2C_2.

12. If the eight points $PQRS\,P_1Q_1R_1S_1$ in Fig. 2 lie on a sphere, then the tetrahedron $ABCD$ is self-conjugate with respect to this sphere.

To prove that the sphere through $APQRS$ passes through the orthocentre of the triangle ABC.

Let O_1 be the centre of the section $PQRS$. AD and BC are conjugate lines with respect to the sphere ; thus if AD meets the section $PQRS$ in L, L is the pole of BC with respect to the section. Thus

O_1L is perpendicular to BC, and AD is perpendicular to BC. So BC is perpendicular to the section ALO_1, and hence if O_1L meets BC in M, AM is perpendicular to BC. If K is the orthocentre of ABC, K lies on AM and $BM \cdot MC = MK \cdot MA$. The triangle LBC being self-conjugate with respect to the section $PQRS$, O_1 is its orthocentre.

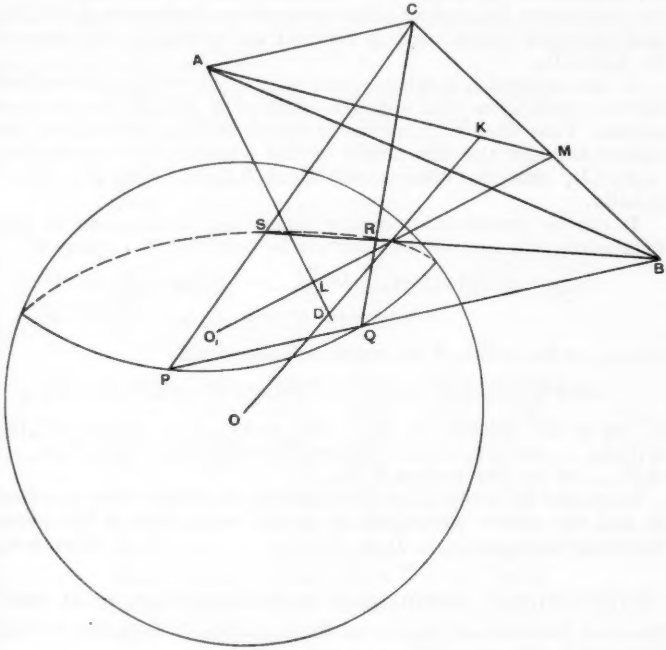


FIG. 3.

Thus $\quad BM \cdot MC = ML \cdot MO_1$

$= $power of M for the section $PQRS$

$= $power of M for the sphere $APQRS$

$= MK \cdot MA.$

Thus K lies on the sphere $APQRS$. Similarly K lies on the sphere $AP_1Q_1R_1S_1$. Similarly the spheres BQQ_1RR_1, BPP_1SS_1, CSS_1RR_1, CPP_1QQ_1 pass through K, the orthocentre of ABC.

In the same way it can be shown that six other spheres can be drawn to pass through the orthocentre of BCD, and similarly for the orthocentres of ABD, ACD.

13. If in Fig. 1 we change the letters P, π_1, π, P into A_3, B_3 C_3, D_3 and take $A_3B_3C_3D_3$ as tetrahedron of reference, and if we take the equation of the plane DBC_1A_1 to be $l\alpha+m\beta+n\gamma+p\delta=0$, then the equations of the eight planes bounding the octahedron $D_2A_1D_1BCA_2$ can be written $\pm l\alpha\pm m\beta\pm n\gamma\pm p\delta=0$.

The octahedron is formed from the three complete quadrilaterals $B_2C_2A_1D_1BC$; $A_2D_2A_1D_1AD$; $A_2D_2BCB_1C_1$, the six diagonals of the octahedron being edges of the tetrahedron of reference $A_3B_3C_3D_3$, and the eight planes touch a conicoid self-conjugate with respect to $A_3B_3C_3D_3$.

If this conicoid is a sphere—that is, if a sphere can be inscribed in this octahedron—the opposite edges of $A_3B_3C_3D_3$ are at right angles. Thus when a sphere can be inscribed in the octahedron, the sphere through the mid-points of the edges of the tetrahedron $A_3B_3C_3D_3$ cuts the spheres $ABCD$, $A_1B_1C_1D_1$, $A_2B_2C_2D_2$ orthogonally.

It can be proved that when a sphere can be inscribed in this octahedron, the octahedron itself will be inscribed in a sphere if

$$\begin{aligned} l^2 : m^2 : n^2 : p^2 = A^2(4\rho^2+b^2+c^2-a^2) &: B^2(4\rho^2+c^2+a^2-b^2) \\ : C^2(4\rho^2+a^2+b^2-c^2) &: D^2(4\rho^2-\overline{e^2+f^2-a^2}), \end{aligned}$$

where ρ is the radius of the inscribed sphere, and

$$-576V^2\rho^2=(b^2+c^2-a^2)(a^2+c^2-b^2)(a^2+b^2-c^2)(e^2+f^2-a^2),$$

V being the volume, A, B, ... the areas of the faces $B_3C_3D_3$, $A_3C_3D_3$, ... and a, b, ... d, ... the lengths of the edges B_3C_3, C_3A_3, ... A_3D_3 ... of the tetrahedron $A_3B_3C_3D_3$.

It can also be proved that the radical plane of the sphere inscribed in and the sphere circumscribed to the octahedron is the plane bisecting the edges D_3A_3, D_3B_3, D_3C_3. R. T. Robinson.

INTERNATIONAL CONGRESS OF MATHEMATICIANS, OSLO, 1936.

The next International Congress of Mathematicians is to be held at Oslo from 13th to 18th July, 1936.

About twenty general lectures will be given; these will provide a general outline of modern mathematics. Those who have already accepted invitations to give such lectures include Ahlfors, Banach, Birkhoff, E. Cartan, van der Corput, Fréchet, Gelfond, Hasse, Hecke, Khintchine, Landau, Neugebauer, J. Nielsen, Ore, Oseen, Siegel, Skolem, Størmer, Veblen, N. Wiener.

Meetings for short communications on recent results are to be arranged under seven sections: Algebra and Theory of Numbers; Analysis; Geometry and Topology; Probability, Mathematical Statistics, Actuarial Theory and Econometrics; Astronomy; Mechanics and Mathematical Physics; Logic and Philosophy, History and Pedagogy.

Plans are being made for trips to the Fjords, Mountains, and the North Cape.

Further information can be obtained from The Secretary (Prof. Edgar B. Schieldrop), International Congress of Mathematicians, Universitetet, Blindern, Oslo.

MATHEMATICAL NOTES.

1165. *Curiosités arithmétiques.*

Gazette, X, p. 43, signale l'égalité

$$\frac{18534}{9267}\times\frac{17469}{5823}=\frac{34182}{5697}$$

dans laquelle les deux termes de chacune des trois fractions contiennent les chiffres 1, 2, 3, 4, 5, 6, 7, 8, 9 pris une fois. Voici d'autres égalitès du même genre.

1.
$$\begin{aligned}&\frac{13458}{6729}\times\frac{13584}{6792}=\frac{13854}{6927}\times\frac{14538}{7269}=\frac{15384}{7692}\times\frac{18534}{9267}\\&=\frac{14586}{7293}\times\frac{14658}{7329}=\frac{15846}{7923}\times\frac{15864}{7932}=\frac{18546}{9273}\times\frac{18654}{9327}\\&=\frac{15768}{3942}=\frac{17568}{4392}=\frac{23184}{5796}=\frac{31824}{7956}.\end{aligned}$$

2.
$$\begin{aligned}&\frac{13458}{6729}\times\frac{17469}{5823}=\frac{13584}{6792}\times\frac{17469}{5823}=\ldots=\frac{13458}{6729}\times\frac{17496}{5832}\\&=\ldots=\frac{17658}{2943}=\frac{27918}{4653}=\frac{34182}{5697}.\end{aligned}$$

3.
$$\begin{aligned}&\frac{13458}{6729}\times\frac{15768}{3942}=\frac{13458}{6729}\times\frac{17568}{4392}=\frac{13458}{6729}\times\frac{23184}{5796}\\&=\frac{13458}{6729}\times\frac{31824}{7956}=\frac{13584}{6792}\times\frac{15768}{3942}=\ldots=\frac{65392}{8174}\\&=\frac{65432}{8179}=\frac{67152}{8349}=\frac{67152}{8394}=\frac{67512}{8439}=\frac{71456}{8932}=\frac{71536}{8942}\\&=\frac{71624}{8953}=\frac{71632}{8954}=\frac{73248}{9156}=\frac{73264}{9158}=\frac{73456}{9182}=\frac{74528}{9316}\\&=\frac{74568}{9321}=\frac{74816}{9352}=\frac{75328}{9416}=\frac{75368}{9421}=\frac{76184}{9523}=\frac{76248}{9531}\\&=\frac{76328}{9541}.\end{aligned}$$

4.
$$\frac{17469}{5823}\times\frac{17496}{5832}=\frac{57429}{6381}=\frac{58239}{6471}=\frac{75249}{8361}.$$

5. On a aussi les égalités

$$\frac{13458}{6729}\times\frac{25496}{3187}=\ldots=\left(\frac{15768}{3942}\right)^2=\ldots.$$

$$\frac{15768}{3942}\times\frac{57429}{6381}=\ldots=\left(\frac{17658}{2943}\right)^2=\ldots.$$

$$\frac{17469}{5823}\times\frac{57429}{6381}=\ldots=\left(\frac{17496}{5832}\right)^3=\ldots.$$

V. Thébault.

1166. *Gamma Functions and Fresnel Integrals.*

1. The integral

$$f(s, a)=\int_0^\infty e^{-ax}x^{s-1}\,dx,$$

where s is positive and $a=p+iq$, is absolutely and uniformly convergent, and therefore continuous, for all values of a such that $p \geqslant p_0 > 0$. In particular, the integral is continuous for $q=0$, when it takes a real positive value.

In the absolutely convergent double integral

$$f(s, a)f(t, a)=\int_0^\infty \int_0^\infty e^{-a(x+y)}x^{s-1}y^{t-1}\,dx\,dy$$

we can make the substitution

$$x=u(1-v), \quad y=uv,$$

which is equivalent to integrating over the area between the axes and the straight line $x+y=u$, and then making u tend to infinity. We thus get

$$f(s, a)f(t, a)=f(s+t, a)\,B(s, t)=f(s+t, a)\,\Gamma(s)\,\Gamma(t)/\Gamma(s+t).$$

It follows that $f(s, a)/\Gamma(s)$ has the exponential property, *i.e.* that

$$\frac{f(s, a)}{\Gamma(s)} \times \frac{f(t, a)}{\Gamma(t)} = \frac{f(s+t, a)}{\Gamma(s+t)},$$

and hence that

$$f(s, a)=\Gamma(s)\{f(1, a)\}^s=a^{-s}\Gamma(s),$$

the exponential being determined by the fact that it is real and positive when a is real.

(The result, of course, is immediately obvious from the point of view of complex function theory ; for the expressions on both sides of the equation are functions of the complex variable a, analytic to the right of the imaginary axis and equal along the real axis.)

The argument holds with hardly any modification if s is complex, its real part being positive.

2. When $p=0$, the integral does not converge absolutely, and only converges at all when the real part of s is less than unity.

If s is real and less than unity, it is easy to see that the integral converges uniformly when p tends to zero, q remaining fixed. The real part and the coefficient of i in the integral are respectively

$$\int_0^\infty e^{-px}\cos qx\, x^{s-1}dx, \quad -\int_0^\infty e^{-px}\sin qx\, x^{s-1}dx,$$

and for either of these we can break up the interval of integration into sub-intervals of length π/q, in each of which the integrand has a constant sign. We thus get in each case an alternating series

$$c_0-c_1+c_2-\dots,$$

the numbers c_n being positive and diminishing towards zero. The

remainder after n terms will be less in absolute value than c_n, and for any given value of n, c_n will have its greatest value for $p=0$, which proves our proposition.

Accordingly we can put $a=-i$, and we get Fresnel's integrals in the form

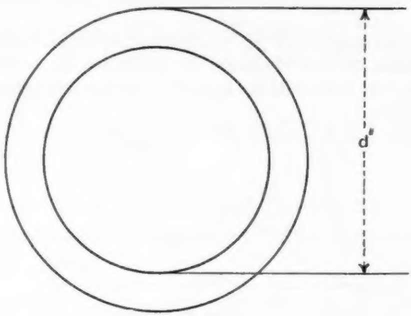
$$\int_0^\infty e^{ix}x^{s-1}dx=e^{is\pi/2}\Gamma(s) \quad \text{for} \quad 0<s<1.$$

M. F. EGAN.

1167. *Approximate weight of an iron pipe.*

The approximate weight in pounds of an iron pipe, l feet long, t inches thick, and d inches in average diameter (that is, diameter of bore plus *one* thickness, as shown in the figure), is

$$10ltd.$$



If t is given in tenths of inches, the formula is merely

$$ltd.$$

Let us see what this gives as the weight per cubic foot, say, x. Then

$$x \,.\, \pi ltd/144=10ltd$$

and

$$x=1440/\pi$$
$$=458{\cdot}4.$$

So the formula would be accurate if iron weighed 458·4 pounds per cubic foot. This is not far from the truth, and any deviation is well within the limits of accuracy of measurement of l, t, d likely to be made.

A. LODGE.

1168. *Theorems on centroids.*

It has struck me that the following theorems in connection with the centres of gravity of simple plane laminae should be added *as standard theorems* to those which replace a triangular lamina by three equal particles, either at the angular points, or at the mid-points of the edges.

1. Irregular quadrilateral replaced by three equal particles.

If J, K are points on the diagonals (which intersect at I) such that $JC=AI$ and $KB=DI$, the centre of gravity of the quadrilateral coincides with that of three equal particles situate at I, J, K.

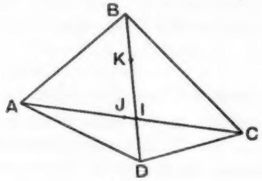


FIG. 1.

2. Trapezium replaced by two particles of masses proportional to partial areas.

Let a, b be the lengths of the parallel sides, h the distance between them. The area can be replaced by particles P, Q of masses proportional to a, b at the centres of gravity of the triangles ABD, CBD.

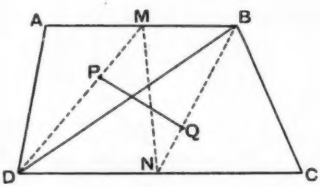


FIG. 2.

Thus taking moments about CD,

$$(a+b)\bar{y}=a\,.\,\tfrac{2}{3}h+b\,.\,\tfrac{1}{3}h,\ \text{etc.}$$

This is slightly simpler than taking particles at the angular points, and also more obvious, as P, Q are actual centres of gravity of the two parts. Also, it lends itself to graphical construction, since the centre of gravity is at the intersection of MN and PQ.

A. LODGE.

1169. *The section of a cone by a plane through its vertex.*

1. The purpose of this note is to obtain the angle between the lines of section by the application of the generalised form of Lagrange's identity; the method followed here can be made more compact by the adoption of tensor notation together with the summation convention.

2. *The generalised form of Lagrange's identity.*

If
$$S=ax^2+by^2+cz^2+2fyz+2gzx+2hxy,$$
$$S'=ax'^2+\dots+2fy'z'+\dots,$$
$$T=axx'+\dots+f(yz'+y'z)+\dots,$$

then $\quad SS' - T^2 = A(yz'-y'z)^2 + \dots + 2F(zx'-z'x)(xy'-x'y) + \dots,$

where $\quad A = bc - f^2, \dots, \quad F = gh - af, \dots .$

3. Let $S=0$ be the cone and $lx+my+nz=0$ the plane; let us so choose new axes $O\xi$, $O\eta$, $O\zeta$ through the vertex that $O\xi$ and $O\eta$ lie in the given plane and have direction cosines (l_1, m_1, n_1) and (l_2, m_2, n_2) respectively. On transformation the lines of section are given by

$$\xi^2 f_{11} + 2\xi\eta f_{12} + \eta^2 f_{22} = 0, \quad \zeta = 0,$$

where
$$f_{11} = al_1^2 + bm_1^2 + cn_1^2 + 2fm_1n_1 + 2gn_1l_1 + 2hl_1m_1,$$
$$f_{12} = al_1l_2 + \dots + f(m_1n_2 + m_2n_1) + \dots,$$
$$f_{22} = al_2^2 + \dots + 2fm_2n_2 + \dots .$$

The required angle θ is given by the equation

$$\tan\theta = 2(f_{12}^2 - f_{11}f_{22})^{\frac{1}{2}}/(f_{11}+f_{22}).$$

By applying the theorem of section 2 and replacing $(m_1n_2 - m_2n_1)$ by $l/(l^2+m^2+n^2)^{\frac{1}{2}}$, etc., we can write the numerator of the right-hand side of the above equation as $2\Sigma^{\frac{1}{2}}(l^2+m^2+n^2)^{-\frac{1}{2}}$,

where $\quad \Sigma = -(Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm);$

again, since

$$l_1^2 + l_2^2 = 1 - \frac{l^2}{l^2+m^2+n^2}, \quad m_1n_2 + m_2n_1 = -\frac{mn}{l^2+m^2+n^2}, \text{ etc.},$$

the denominator $(f_{11}+f_{22})$ becomes

$$(a+b+c) - f(l, m, n)/(l^2+m^2+n^2),$$

where $\quad f(l, m, n) = al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm.$

Thus we can write

$$\frac{\cos\theta}{(a+b+c)(l^2+m^2+n^2) - f(l, m, n)} = \frac{\sin\theta}{2\Sigma^{\frac{1}{2}}(l^2+m^2+n^2)^{\frac{1}{2}}}.$$

S. SUBRAMANIAN.

1170. *The integration of* sec x *and* sec³ x.

If $\quad \sec x = \cosh v,$

then $\quad \sec^2 x - 1 = \cosh^2 v - 1$

and $\quad \tan x = \sinh v,$

so that, since $\sec x \tan x\, dx = \sinh v\, dv,$

we have $\quad \sec x\, dx = dv$

and
$$\int \sec x\, dx = \int dv = v$$
$$= \arg\cosh(\sec x)$$
$$= \log(\sec x + \tan x).$$

Also,
$$\int \sec^3 x\, dx = \int \cosh^2 v\, dv$$
$$= \tfrac{1}{2}\int (1+\cosh 2v)\, dv$$
$$= \tfrac{1}{2}(v + \tfrac{1}{2}\sinh 2v)$$
$$= \tfrac{1}{2}\{\arg\cosh(\sec x) + \sec x \tan x\}.$$

S. RICHARDSON.

REVIEWS.

Variationsrechnung und partielle Differentialgleichungen erster Ordnung. By C. Carathéodory. Pp. xi, 407. Geb. RM. 22. 1935. (Teubner)

The characteristic feature of this book is a balance between the Hamilton-Jacobi and the Euler-Lagrange methods. If one chooses to describe the integrand of a curvilinear variational problem as a kinetic potential one is doing dynamics, Euler's equations being nothing else than Lagrange's equations of motion. So it is clear *a priori* that the methods of Hamilton and Jacobi are applicable to the calculus of variations. That is to say, one calculates the components of velocity in terms of the generalized momenta and substitutes them in the total energy, thus obtaining the Hamiltonian function. The Hamiltonian equations of motion are then seen to give the characteristics of a certain partial differential equation of the first order, commonly called the Hamilton-Jacobi equation.

In the volume under review these methods receive full recognition. Indeed, the first part of the book (pp. 1-163) is devoted to topics relevant to these equations. The first four chapters describe, among other things, the characteristics of a single equation and of systems of equations, Lagrangian and Poisson brackets, and systems (of functions or equations) in involution. There follows a chapter on tensor calculus, and Chapters VI-X are on canonical and contact transformations (the former are the most general transformations, usually included among the latter, which leave the form of the Poisson brackets unaltered), the problem of Pfaff, function groups, and on various methods of integrating the partial differential equations which have been considered.

These matters are described clearly and thoroughly, and this first part, taken by itself, is a welcome addition to the literature. It seems a pity that the subject of integral invariants has been omitted. In the introduction the author explains that, among other topics, they were excluded in order to bring the book down to a reasonable price. All the same, a few sections, or a short chapter, on integral invariants would have completed an admirable account of this most satisfying branch of mathematics. They are intimately related to the Hamilton-Jacobi equations and have a first-class application to mechanics in the theory of adiabatic invariants. As it is, the book contains a certain amount of detail which, though good in itself, could, I think, have been sacrificed with profit.

The second part, after a preliminary chapter on quadratic forms, proceeds with a penetrating account of small variations, in which the author breaks away from the traditional presentation. The search for extremals is based upon certain differential equations (the " Fundamental equations ", p. 201) from which one can, by elimination, derive the Hamilton-Jacobi equations, expressed in terms of the Lagrangian kinetic potential. Any solution of the latter represents a family of hyper-surfaces whose transversal trajectories (the generalization of orthogonal trajectories) constitute a field of extremals. Thus the field in question is what is often called a Mayer field (the generalization of a normal congruence in Riemannian geometry), and the differential of the solution is the integrand of Hilbert's line integral.

The fundamental equations are immediately followed by a section on the Weierstrass E-function, in which it is shown that, subject to the regularity condition, any extremal in such a field provides a strong minimum. There follow sections on the Hamiltonian function and canonical coordinates, and then the Euler equations are introduced as the Lagrangian equivalent of the Hamiltonian equations.

Chap. XIII describes the problem in parametric form, the integrand being positively homogeneous of the first degree, Chap. XIV describes certain special cases, and in Chap. XV there is a discussion of the case where the integrand is quadratic in $\dot{x}^i$. I think that Chap. XIII falls away from the high standard of its predecessors; for the methods and results of Chap. XII are applied in such a way as to obscure, rather than exhibit, the mechanical interpretations which have, up till then, been so brilliantly suggested. A similar remark applies to the geometrical interpretations; for Chap. XIII can be regarded as an introduction to Finsler geometry, which is one of the discarded topics already alluded to. As far as this book is concerned, the step from Riemannian to Finsler geometry is trivial. One need only remark that the functions g_{ij} of the former may be allowed to depend on the variables dx^i, in which they are homogeneous of degree zero, and a single formalism does for both. Beyond the fact that the relation of orthogonality is, in general, asymmetric, one need not take the distinction seriously until one considers covariant differentiation and displacements, on the geometrical side, and, on the analytical side, directions for which the regularity condition breaks down. So $L=\frac{1}{2}F^2$ is surely the natural link between Chapters XII and XIII, as it is between XII and XV (see p. 256), and I think Chapters XIII and XV could profitably have been combined, perhaps after a short section on the Finsler metric. If I have laboured this point, it is largely because writers on the calculus of variations have constantly ignored Finsler's ideas, even though they have been willing to treat the special case of Riemannian geometry at some length.

After giving examples of boundary value problems, Chap. XVI settles the question of the absolute minimum for regular, positive, boundary value problems, both in the small and in the large, Chap. XVII contains an account of periodic extremals, and the final Chap. XVIII an account of the problem of Lagrange and related problems. The existence of the absolute minimum is extended from the small to the large by Hilbert's method. Periodic extremals fall under Poincaré's influence, but after this short, though very welcome, digression, the book concludes in the Hamilton-Jacobi tradition.

In restoring this tradition to the calculus of variations the author has done a great service to mathematics. From the journalistic point of view, any possible demerits have a high scarcity value. This, and a reason already given, account for the energy spent in criticizing what is a grand book. Section 232, containing the " fundamental equations ", is a masterpiece.

J. H. C. W.

Mathematical Problems of Radiative Equilibrium. By E. Hopf. Pp. viii, 105. 6s. 1934. Cambridge Tracts, 31. (Cambridge)

In this tract Dr. Hopf presents a more or less complete account of the integral equations which occur in the theory of the radiative equilibrium of the stellar atmospheres. Though the monograph is decidedly mathematical in its outlook, it should yet prove indispensable to the serious student of astrophysics who wishes to have at his command all the mathematical resources available for the treatment of his problems.

The physical problem which underlies the discussion is to determine the distribution of temperature and light in material stratified in parallel planes with or without radiation incident on it from outside. The most elementary problem of this character was first formulated by Schwarzschild, who considered the radiative equilibrium of material in local thermodynamical equi-

librium (*i.e.* Kirchoff's law holding locally). For this problem the equation of transfer is

$$\cos\theta \frac{\partial I(\tau,\theta)}{\partial\tau} = I(\tau,\theta) - J(\tau),$$

where
$$J(\tau) = \frac{1}{4\pi}\int I(\tau,\theta)\,d\omega = B(\tau) = (\sigma/\pi)\,T_\tau^4.$$

In the above equations $I(\tau,\theta)$ is the specific intensity of radiation at optical depth τ and in a direction making an angle θ with the normal, and $B(\tau)$ is the integrated intensity of radiation at τ given by the Stefan-Boltzmann law. The problem is to determine $I(\tau,\theta) \geqslant 0$, $\tau < \infty$ when the radiation $I(0,\theta) \geqslant 0$, $\theta > \pi/2$ is incident on the surface $\tau = 0$. If $I(0,\theta) = 0$, $\theta > \pi/2$, then $J(\tau)$ is found to satisfy the integral equation

$$J(\tau) = \frac{1}{2}\int_0^\infty J(t)\,E_1(|\tau - t|)\,dt,$$

where
$$E_n(x) = \int_1^\infty e^{-sx}s^{-n}\,ds.$$

There are correspondingly other equations for the other physical problems (which Dr. Hopf classifies into four groups). These equations are set up in Chapter I. It is unfortunate that Dr. Hopf did not include in this chapter an account of Hilbert's proofs of the fundamental laws of radiation, for it was Hilbert who first introduced the theory of integral equations in the treatment of radiative problems.

In Chapter II the general existence and uniqueness theorems for the Schwarzschild problem are established. The properties of asymptotically linear solutions are also considered here. An interesting outcome of the analysis of this chapter is the *exact* relation which is found to exist between the effective temperature and the boundary temperature, namely, $T_e^4 = (\sqrt{3}/4)\,T_0^4$—a remarkable relation discovered independently by Dr. Hopf and Dr. Bronstein. Chapter III deals (less completely) with purely scattering atmospheres—the Schuster problem. Chapter IV deals with some general integral equations, the special cases of which are those that occur in the theory of radiative equilibrium. Chapter V initiates a preliminary discussion of those radiative problems which lead to non-linear integral equations. Finally, it also contains an account of Milne's treatment of the planetary nebula.

The general arguments in the book are clearly set out, and the mathematics can be followed without much difficulty. The reviewer, however, feels that the arrangement of the material could be much improved and that a little more stress could have been laid on the physical situation. Thus in the treatment of the planetary nebula the essence of the physical situation (namely, that there are two superposed fields of radiation—the field of the ultra-violet radiation and the field of Lyman-α radiation—and that there is a continual " conversion " of the one type of radiation into the other) is entirely overlooked. Approximate treatments of this last problem have been given, but the rigorous mathematical treatment on the lines of Dr. Hopf's tract still remains to be worked out. Again, it appears that in the future one would have to take properly into account the curvature of the outer layers of a star in solving the problems of radiative equilibrium, and in the treatment of these and other problems Dr. Hopf's tract must be a sure and valuable guide.

S. Chandrasekhar.

Interpolatory Function Theory. By J. M. Whittaker. Pp. 107. 6s. 6d. 1935. Cambridge Tracts, 33. (Cambridge)

It is as well that a review should begin at the beginning, and so I start by

endorsing the statement, made by the author in his preface, that "the title of this work is not, perhaps, a very happy one, but I cannot think of a better." Neither can I think of a better way of handling his subject-matter.

This little book is a "Cambridge Tract" only in the sense that it is one of the works published in that excellent series. Its excellence, for let there be no doubt about the very high quality of the work, is not that usually associated with the Cambridge Tracts. It is not the usual thorough development of one topic "which can be treated adequately within the limits of 100 pages". It is rather an account of *some* of the things that join up two topics—here, the theory of integral and meromorphic functions, and the theory of interpolation formulae—neither of which can be "treated adequately within the limits of 100 pages".

The result is an entertaining half-dozen chapters on topics which, for the most part, have been developed within the last ten years or so. Chapter I deals with the general question of expanding a function in terms of a given "basic" set of polynomials; its chief weapons are elementary facts in matrix theory and orthodox convergence properties. Perhaps the most illuminating way of giving the reader an idea of the rest of the book is to give the chapter headings and to state what seems to me the most interesting result contained in each chapter.

Chapter 2, The sum of a function; "If $f(z)$ is an integral function of given order, there is an integral function $g(z)$ of the same order such that

$$g(z+1)-g(z)=f(z)\text{''}.$$

Chapter 3, Properties of successive derivatives; "Let $f(z)$ be a meromorphic function and let E denote the set of zeros of $f(z), f'(z), \ldots$. Then a point z belongs to E', the derived set of E, if and only if it is equidistant from the two poles of $f(z)$ which are nearest to it".

Chapter 4, Interpolation at the integers; "Cardinal series interpolation is 'consistent'" is a pleasing result on the formal side of interpolation, while Pólya and Hardy's "2^z is the smallest transcendental integral function which takes integral values at $z=0, 1, 2, \ldots$" is a striking result which indicates very well a type of work which is given prominence throughout most of the book.

Chapter 5, Interpolation at the lattice points; "If $f(z)$ is an integral function which is bounded at the lattice points and $\log M(r)/r^2$ tends to zero as r tends to infinity, then $f(z)$ is a constant". When I read Pólya's proof of that I was green with envy; it has the perfect timing of a late cut by a master batsman.

Chapter 6, Asymptotic periods; "The single periodicity of $\sin z$ and the double periodicity of $\operatorname{sn} z$ appear at first sight to be phenomena of much the same kind; but if we enlarge the concept by including asymptotic periodicity it becomes evident that they belong to classes of phenomena which differ in almost every respect".

It is no mean achievement to have worked into a connected whole the topics which here appear; and it is all the more gratifying to find that it has been done in a manner which maintains the standard of thorough, scholarly work expected of a Cambridge Tract.

The work is not, of course, as complete as a tract on a more limited subject would be; but it is, in my opinion at least, one of the most interesting and lively volumes in the series.

Finally, one small grumble. I do not object to such phrases as "it is obvious from (2·1)", but oh how I hate "it is obvious" when I begin by looking at "it" from the wrong angle.

W. L. F.

Il Passato e il Presente delle principali Teorie Geometriche. Storia e Bibliografia. By G. Loria. 4th edition. Pp. xxiv, 468. 1931. (Cedam, Padoa)

This book is in two parts, the first formed of the substance of the last edition, and ending with the nineteenth century, the second prolonging the account by thirty years. The subject is divided into eleven branches, to each of which are devoted a chapter in the first part and a section in the second. The book is not intended to be a readable or even an intelligible story of the development of geometry. There are many descriptive paragraphs, but essentially the volume is a comprehensive and uncritical bibliography. The author calls it a bibliographical guide; but it is hardly that: a guide ignores what is trivial in order the better to point out what is significant, and transmits the benefit of general experience or of an individual judgment.

One of the most difficult problems of compilation is to discover relevant work under an irrelevant title. We all know that a bibliography of vector analysis is incomplete without a reference to Heaviside's *Electro-Magnetic Theory*, and that the literature of the theory of equations includes Klein's *Ikosaeder*, but in less notorious cases the compiler is at the mercy of chance. Prof. Loria does not record, for example, that there is more geometrical work with Grassmann's calculus in A. N. Whitehead's *Universal Algebra* (I, Cambridge, 1898) than in many of the memoirs of which he gives the titles, or that the geometrical section of B. A. W. Russell's *Principles of Mathematics* (I, Cambridge, 1903) is far more important than his *Foundations of Geometry* (Cambridge, 1897).

" It is fortunate ", wrote Greenstreet of the third edition of this book, " that those to whom the book will be of use will be no more than irritated by the appalling frequency of misprints ", and I can only echo the gentle reproof. My eye was caught by Stawel Ball, Coodlige, Gallatley, Heavood, Hollkroft, Lefschwetz, Richemond, Roussel, Schaw, Wodd, Wrigt, and Crisholm Young; it is to be hoped that future historians will not take Prof. Loria's word as proof of the existence of Hilda Hilton. There are several variations on Plücker, and the reader who assumes that at least Italian names are immune will be forced to cry out Eurique! It is not only proper names that suffer. Prof. Loria has many English friends, and I am sure he could find one who would be proud to read proofs for him.

In the index different writers with the same name are usually distinguished, but two of the most familiar English names are overworked, for the four entries under Roussel refer to three Russells, while one Wren is Christopher and the other is T. L.; one paper ascribed in the index to Rouse Ball is ascribed correctly in the text to R. S.

Although this is a book to which the classical advice on references most emphatically applies, it is one which every mathematical librarian finds invaluable. But for the sake of the student I wish that Prof. Loria instead of giving us in a catalogue the evidence of his immense industry had given us the benefit of his ripe judgment in the discriminating guide which he has been too modest to write. E. H. N.

Les Logarithmes et les Puissances. By H. Tripier. Pp. viii, 50. 8 fr. 1934. (Vuibert)

So far as it is safe to draw conclusions from this individual book it appears that the method which it recommends and uses for the elucidation of the logarithmic and exponential functions is little used in France. In this country it has been familiar to mathematicians at least since the appearance (1908) of Professor Hardy's *Pure Mathematics*. It was very strongly advocated by

Klein (*Elementarmathematik*, 1908), and is probably becoming generally familiar to school teachers through the books of Messrs. Durell and Robson.

The book itself is a workmanlike exposition and contains a chapter on the historical development; the criticisms which follow are due rather to the inherent interest of the subject than to actual defects, except as regards the historical section.

The book is not primarily intended for mathematical specialists, but rather for the ever-growing number of those who in scientific work of all kinds meet these functions. The author holds that intelligent acquaintance with them ought to be widespread and therefore has sought and found what he thinks the simplest way of dealing with them.

The primary question as to method is whether to follow Euler and take the exponential as the direct function and the logarithm as its inverse; or to define $\log x$ as $\int_1^x \frac{dx}{x}$ and the exponential as its inverse. The subordinate question is whether to use $\int_1^x \frac{dx}{x}$ frankly and arithmetically or to replace it by the relevant area under the arc of a hyperbola.

The author adopts the last course and claims that it " renders evident, with a guarantee of absolute certainty, the existence and continuity of the two functions, for all real values rational or irrational of the independent variable. . . . The sight of the curve easily sustains the knowledge of those functions and of their principal properties ".

But, as he says, " it is delicate and long to establish this existence and this continuity starting from the arithmetical definition ". This surely means that, to the pure mathematician familiar with the difficulties involved in the word " continuity ", the geometrical treatment is not satisfactory.

But even if his claim " a guarantee of absolute certainty " is open to criticism by the pure arithmetician, it is justified so far as the ordinary student is concerned. Klein is worth quoting: " The convincing power which dwells in such path-breaking considerations is naturally very different for different individuals. Many—and *among them I count myself*—feel themselves extraordinarily satisfied thereby; others again who are one-sidedly disposed to the purely logical side, find them completely empty and cannot imagine how anyone can adopt them as the fundamentals of mathematical treatment. And yet considerations of this kind have often been the beginning of new and fruitful methods of treatment " (*op. cit.* p. 227).

But geometrical treatment in fact hides the difficulties of " continuity ", and the author in adopting them (rightly, as we consider) might as well have been consistent and ignored irrationals. He gives conscientiously the definition of a^X when X is irrational; in the circumstances this seems needless and unprofitable for the readers he has in mind. We prefer the view of Durell and Robson (*Advanced Trigonometry*, p. 64): " A discussion of the theory of irrational numbers is beyond the scope of this volume; we shall not, therefore, at this stage define the function e^y for irrational values of y ". In fact, all measurements are necessarily approximate, and, for the scientific worker as against the philosopher and pure mathematician, irrationals do not effectively exist.

The least satisfactory part of the book is the historical chapter, and as points of real interest are involved perhaps some remarks on this may be permitted, especially as the defects are rather those of the writers on whom the author relies than his own, and the misunderstandings are widespread.

The author has evidently no first-hand acquaintance with Napier's work, and one can only infer that his authorities have as little. Even Klein fails to

appreciate the subtlety and completeness of Napier and the way in which his work excels that of Bürgi (*Elementar*, pp. 160, 161, 187).

Napier, as the author says, like Bürgi, took as his working basis the correspondence between an arithmetic and a geometric series. But behind this, as the author does not say, Napier had a perfectly clear idea of a continuous definition. His fundamental idea is of two points moving along lines, the one with uniform velocity, the other with velocity *always* proportional to its distance from the remote end of the line: " Whence ", he says in one of his statements (Constructio 25), " a geometrically moving point approaching a fixed one has its velocities proportional to its distances from the fixed one ".

When it comes to calculation, whether in Napier's form or in the ordinary numerical evaluation of a definite integral, discontinuity is inevitable; but Napier's fundamental definition is as continuous as that based on the hyperbola.

The author refers (p. 41) to doubts as to whether Napier saw the analogy between his logarithms and the hyperbolic areas. It would have been remarkable if he had done, for coordinate geometry was not invented till a quarter of a century after his time!

But the most important omission is that of any reference to the fact that Napier's definition is really identical with $\log x = \int_1^x \frac{dy}{y}$.

The omission is not surprising for the fact seems even now to be little known, so it may be worth while to demonstrate it again.

Put into its simplest form Napier's definition comes to this:

Let AB be a line of unit length and let a point starting at A move towards B with velocity always numerically equal to its distance from B; then the time it takes to reach a position P is the logarithm of BP;

i.e.
$$\log y \equiv t = \int_y^1 \frac{dy}{y}.$$

Not merely is this use of velocity basic in Napier's work, but his method of interpolation depends upon it: if $\log N$ is known and that of $N+h$ is required, he argues that the velocities in the interval lie between N and $N+h$, *i.e.* the correction required to obtain $\log(N+h)$ lies between $\frac{h}{N}$ and $\frac{h}{N+h}$.

It is again the initial use of this idea which differentiates Napier's logarithms from those of Bürgi. (This is the point at which Klein is defective.)

Napier's logarithms are not logarithms to a base $1-10^{-7}$, and the numbers in his " Radical Table " are not powers of $1-10^{-7}$ nor of any other single factor. He uses three factors in all, viz. $1-10^{-7}$, $1-10^{-5}$, and $1-{\cdot}0005$. The results he puts into their proper places in the logarithmic scale by his method of interpolation.

Further, the basic logarithmic interval is not 10^{-7} but is deliberately chosen as the best approximation to the ideal interval. He argues: it lies between 10^{-7} and $\frac{10^{-7}}{1-10^{-7}}$ so as " best of all " I take the mean of these extremes.

In short, reduced to modern terminology all the following ideas are implicit in Napier's work:

1. $\log x = \int_1^x \frac{dy}{y}$.

2. $\frac{d}{dx} \log x = \frac{1}{x}$.

3. Antilog 1 lies between $\left(1+\frac{1}{n}\right)^n$ and $\left(1+\frac{1}{n}\right)^{n+1}$.

4. The gradient of the curve of uniform growth ($y=e^x$) is numerically equal to the ordinate: $\frac{d}{dx}e^x=e^x$.

Thus what may be termed the twentieth-century method is in fact a return to Napier.

It may well be asked why Napier's method, perfectly sound in principle and convenient for calculation, was ever abandoned. The reason seems to be that the later seventeenth century (Mercator and Newton) used the newly discovered infinitesimal calculus to develop $\log(1+x)$ and e^x in series. This yielded an even more convenient method of calculation, but the methods were of course crude, and as Klein says (*op. cit.* p. 163), " the productive epoch was followed by the critical, I may almost say the period of moral distress, in which the attempt was made to base the newly discovered results firmly and to rid them of anything possibly false ". But it was not till the early nineteenth century (Cauchy) that " complete mathematical exactitude was given to the theory of the logarithm ", and those who were brought up on the ordinary English textbook of the late nineteenth century (Todhunter) will remember how very unsatisfactory the treatment there still was.

To return to the book before us, we may note that the author (p. 20) enumerates six properties of the logarithmic function. With five of these anyone who has used tables with any degree of intelligence will be perfectly familiar, and the other is simply that the function has a derivative which is the inverse of the argument. A non-mathematical reader might perhaps wonder why he should be invited to study a book which added so little to his knowledge!

But the author adds a remark of real value to those to whom the book is specially addressed: the fact that the differential of the logarithm is the ratio of that of the variable to the variable itself, *i.e.* is the unitary variation, has as its consequence the constant appearance of the function in the quantative study of Nature.

As Klein says (*op. cit.* p. 85) in the ordinary school treatment $\sin x$ and $\cos x$ appear in the geometry of the triangle, $\log x$ as a convenience for numerical calculation; why do these appear in the most various domains which have nothing to do with geometry or with the technique of calculation? $\sin x$ plays a central part wherever vibrations are in question; e^x expresses the " law of organic growth ". Both result from differential equations of the simplest type $\left(\frac{d^2y}{dx^2}=\pm y\right)$ which lie at the root of all those applications. The natural basis for the study of both as functions therefore lies in the quadrature of simple curves.

W. C. F.

Hypocycloïdes et Epicycloïdes. By J. Lemaire. Avec une Préface de M. d'Ocagne. Pp. viii, 295. 22 fr. 1929. (Vuibert)

The hypocycloids and epicycloids have a wealth of metrical properties, and M. Lemaire's aim, like Proctor's long ago, is to develop these by the methods of Euclidean geometry. Only he starts differently. Proctor defines the curves in the usual way as roulettes, but M. Lemaire uses the tangential definition: if two points describe a fixed circle with constant speeds, the envelope of the line joining them is a hypocycloid or an epicycloid according as their directions of motion round the circle are the same or opposite. The point of contact is

readily identified, and M. Lemaire's accounts of the curves unfold with the deceptive inevitability that betokens a writer who is completely master of his subject. But note the plural. The definition covers both kinds of curve, the basic distinction being one of sign alone, but M. Lemaire gives separate short chapters on the general hypocycloid and the general epicycloid, with only occasional references from the later chapter to the earlier for details of proofs; the continuity and independence of each chapter is secured at the cost not merely of a great deal of repetition but also of the omission of all parallelisms and contrasts. Similarly the sections on particular curves include paragraphs that vary in only the most trivial manner from one curve to another; the results are individual, but nothing except the results is being learned by their accumulation.

The hypocycloids with three and four cusps, the epicycloids with one, two, and three cusps, and the two degenerate cases, the cycloid and the involute of the circle, all receive some attention, but half of the book is devoted to the first of these special curves. It is well known that the envelope of the Wallace-Simson lines of any triangle is a three-cusped hypocycloid, and once this curve is accepted as a familiar figure, a new chapter is begun in the geometry of the triangle. Other interesting problems concern conics which have multiple contact with a three-cusped hypocycloid, and simple Cremona transformations connecting a three-cusped hypocycloid with a circle. It is somewhat surprising that a curve which was studied by Steiner, Laguerre, and Cremona has not been the subject of an independent treatise, and M. Lemaire's book, of which a copy has at last reached us, is welcome as coming near to filling this gap in geometrical literature. E. H. N.

Premier Livre du Tétraèdre. By P. Couderc and A. Balliccioni. Pp. viii, 204. 40 fr. 1935. (Gauthier-Villars)

A textbook on the Tetrahedron has long been overdue, and we have here a praiseworthy attempt to supply the need. As far as one can judge from this first volume (no contents list is given of the second), the standard is meant to be that of the Scholarship class in secondary schools. The various types of tetrahedron are treated in detail, and many formulae are given concerning the orthocentric and equifacial, as well as the general figure. There is also a full discussion of the related spheres, including those tangent to a skew quadrilateral. (No coordinate geometry is used in the book.)

It is evident, from the above synopsis, that the authors have been at pains to present a methodical and systematic treatment of the subject, but there is, to our mind, a serious omission in that no use is made of vector algebra; so many of the formulae become automatic when expressed in terms of scalar and vector products. If we take, for example, the edges DA, DB, DC as the vectors α, β, γ, and if P is the point whose tetrahedral coordinates are (x, y, z, t), then we have at once $\overline{DP}=x\alpha+y\beta+z\gamma$. If P' is (x', y', z', t'), then $\overline{PP'}=(x'-x)\,\alpha+(y'-y)\,\beta+(z'-z)\,\gamma$, so that expressions for PP'^2, for the perpendicularity of two lines, and so forth, can be written down at once.

Clearly much play can be made with the foregoing principles, and there is no reason why the boy in the Mathematical Sixth should find difficulty with the amount of vector lore required. For the orthocentric tetrahedron, in particular, the use of vectors not only makes the work exceedingly simple and suggestive, but enables the properties to be instantaneously extended to similar configurations in higher-dimensional space. Of course, it may be intended to adopt this point of view in the second volume. H. L.

Exercices d'Analyse. IV. Équations aux dérivées partielles du premier ordre. By G. JULIA. Pp. 230. 60 fr. 1935. (Gauthier-Villars)

In this fourth volume Professor Julia continues the general plan so admirably carried out in the earlier parts of his valuable *Exercices*. Assuming that his readers are tolerably familiar with the more elementary theory and machinery of partial differential equations, as given, for example, by Goursat, he aims partly indeed at increasing their powers of manipulation but much more at strengthening their grasp of ideas and furthering their penetration into the real meaning of the subject. It is one which beginners frequently find difficult; for in coming to it from ordinary differential equations they find the partial equations less clear-cut and much less concrete. To counteract this feeling Professor Julia, with all the expository skill we expect from him, stresses the geometrical aspect of the subject; frequently he gives two solutions to a problem, one with an analytical and the other with a geometrical bias, and by comparing the two the reader should find it much easier to comprehend the essential content lying behind the algebraic façade.

Of the 55 problems propounded and solved, most are taken either from papers in calculus at the Sorbonne or from Goursat. The whole set of volumes is of tremendous assistance both to those who are learning and to those who are teaching analysis. We look forward with pleasure for the appearance of the remaining part to complete a work of which the utility and interest cannot be too highly praised. T. A. A. B.

Esercizi di Analisi infinitesimale. By G. VIVANTI. 3rd edition. Pp. 404. L. 55. 1935. (Lattes, Turin)

Professor Vivanti's book is a valuable complement to the usual Continental textbook on the calculus. It contains 573 examples, with solutions either fully worked or, in a few cases, sketched, distributed as follows: preliminaries on limits, continuity, etc. (14); derivatives and integrals of functions of one variable (233); derivatives and integrals of functions of several variables (35); geometrical applications (99); differential equations (178); calculus of variations (14). References are given to the author's *Lezioni di analisi matematica*.

The section on linear differential equations with constant coefficients suggests that the standard way of obtaining the complete solution is by "variation of parameters", which seems rather ponderous to be applied, for example, to No. 504:

$$y''' - 3y' + 2y = x^2.$$

In this country, however, we are perhaps too contemptuous of this method. It is good to see that in dealing with partial differential equations a fair show is given to equations in more than two independent variables.

A student reading this book should find both his manipulative powers and his respect for fine points increased. T. A. A. B.

Höhere Mathematik für Mathematiker, Physiker und Ingenieure. III. By R. ROTHE. Pp. ix, 238. RM. 6.60. 1935. Teubners mathematische Leitfäden, 23. (Teubner)

This volume completes the text of Dr. Rothe's manual, the fourth volume being devoted to further examples and solutions.

Two sections of this volume deal with curvilinear coordinates, and curvilinear and multiple integrals. The remaining section, filling roughly half the book, is on differential equations. The general plan of cementing theory and practice with a large and interesting collection of illustrative examples is

carried on throughout the book, with results which are, we think, particularly notable when we come to the differential equations. Here, although only 114 pages altogether are available, we have 30 pages of introductory matter before methods of integration are mentioned. In these 30 pages, the construction and importance of the subject are exhibited by illustrations from dynamics, physics, ballistics, electrical theory, " looping the loop ", and geometry ; but this does not mean that the author thereby considers himself exempt from the duty of dealing with the abstract theory. No sooner has he discussed the elementary methods of integration applicable to equations of the first order than he turns to the existence theorems, and gives a simple and lucid account which ought to be intelligible to anyone capable of understanding what the problem really is. Linear equations are treated concisely but adequately, save that the only method employed in the determination of particular integrals is that of variation of parameters. With a little care in presentation this method can be made to appear very natural, but to use it on quite simple constant-coefficient equations is as heavy as the exclusive employment of operational methods can be. Whatever may be true of the teaching of mathematics for technical applications, the mathematician in the making must have training in the selection of appropriate tools.

We congratulate Dr. Rothe on the completion of the main part of his treatise. Its clarity of expression and wealth of illustration should make it a valuable aid to students and teachers, whether their interests are theoretical or practical. Considering the neat format and excellent paper and printing, as well as the amazingly large amount of material included, the book is splendid value for the money. T. A. A. B.

A Complete School Algebra. By A. Ritchie-Scott. Pp. 711, including answers, 8s. 6d. ; without answers, 7s. 6d. Part I. With answers, 5s. ; without answers, 4s. Part II. With answers, 5s. ; without answers, 4s. 1935. (Harrap)

This is an arresting work, at once stimulating and perplexing.

It is offered to the world with the benediction of Professor E. T. Whittaker. That commands immediate respect ; but as Professor Whittaker is careful to say that his appreciation is that of a university professor, the schoolmaster may, without undue diffidence, criticise the suitability of the exposition for young pupils.

As to the title, the word " complete " must not be taken literally. Except for a chapter on the Multinomial theorem with some reference to partitions, the book closes with the Binomial theorem for a positive index. And as a school algebra it presents unusual features. In delaying the simple equation to Chapter VIII, it seems to react to a procedure now generally discredited. But an examination of the earlier chapters will show that this is only apparent ; indeed the author introduces much original matter and method. He looks for a maturity and responsiveness in young pupils that schoolmasters would be cheered to find. And Chapter VII preceding Chapter VIII looks like a very large cart pushing a diminutive horse.

Even so, this may represent the author's experience tried out on average beginners. If so, one would welcome an assurance to that effect and, stimulated by a courageous example, run one's teaching on bolder lines.

The book's scholarship is everywhere apparent, and offers many interesting ideas rare in elementary instruction.

Again, while the author in places seems to aim at widening the pupil's outlook and stimulating his powers, in others he elaborates a topic with an

unusual intensity of purpose. For example, the discussion of the quadratic function and its graphical treatment seem to aim at such a thorough mastery of

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

as is only needed by scholarship pupils doing analytical geometry.

Some mention of particulars of earlier chapters will exemplify these remarks. The first batch of exercises asks the pupil to translate into formulae statements of various laws, such as those for electromotive force, Young's modulus, etc.

Chapter III, which deals with surds, includes the approach to $\sqrt{7}$ through the convergences 2, 2·6, 2·64, ... to 2·6457513 and 3, 2·7, 2·65, ... to 2·6457514. This is followed by a graphical representation of incommensurable numbers. And in the same chapter, the pupil sees

$$\frac{2\sqrt{(3\cdot 1)}}{2\sqrt{(2\cdot 4)}} \text{ simplified to } \frac{\sqrt{(7\cdot 74)}}{1\cdot 6}.$$

Chapter IV introduces graphs through linear arrangements and scales.

Chapter V on multiplication and factorisation devotes a number of pages to the identity $(a+b)(a-b)\equiv a^2-b^2$ and concludes with the question

$$\text{“ Simplify } \frac{(a-b+c)^2-(a-b-c)^2}{4(x+y)} \div \frac{a(b+c)}{x-y}.\text{”}$$

In Chapter VII the pupil is solving equations such as

$$\frac{1}{x}=\frac{1}{u-v}+\frac{1}{u+v},$$

and problems such as

" A train runs between A and B, a distance of s miles. The scheduled time for the journey is t hours, but in going from A to B it takes h hours longer than this, making up for the loss on the return journey by going from B to A in h hours less than the scheduled time. (i) Find the average value of the two actual velocities. (ii) Find the average velocity of the train over the double journey.

Explain and reconcile the two answers."

Then follows Chapter VIII entitled " Simple Equations ". And the word " simple " here acquires an added significance.

It is difficult to imagine that this can be a suitable book for the lower classes of schools, but the teacher who consults it will be well repaid for the perusal.

F. C. B.

A School Certificate Algebra. By W. G. Borchardt. With answers. Pp. xii, 431, lxxvi. 4s. 1935. (Rivingtons)

Borchardt's textbooks have been widely used and appreciated for more than thirty years. A popularity of this kind may imply that in spite of their acknowledged merits they are now behind the times. The present work is free from that reproach. The author explains in the Preface that he has kept an eye on the latest report of our Association, from which he has adopted many suggestions, and that much new matter has been introduced.

Chapter I introduces Algebra as generalised arithmetic, and Chapters IV and V deal with the first Four Rules, powers, roots, H.C.F. and L.C.M. from this point of view.

Chapter II deals with formulae, and Chapter III with the simple equation. Thus in 32 pages the pupil becomes acquainted with the threefold basis of the subject.

Part I proceeds thereafter on what are now regarded as normal lines as far

as simultaneous equations of the 1st degree in three unknowns and of the 2nd in two unknowns and the Remainder Theorem.

Part II includes Progressions with a section on Annuities, Theory of Quadratics, Theory of Logarithms, Variation, Partial Fractions and Graphs. The graphical section has a technical bias and deals with gradient, the area under the graph and the law of the graph in such a case as $yx^n = c$.

Examples are numerous and varied, and there are frequent batches of revision examples. There are also XCIV Test Papers.

The requirements of various certificate examinations are more than amply met. F. C. B.

Elementary Practical Mathematics. By E. W. GOLDING and H. G. GREEN. II (Second Year). Pp. xii, 188. 1934. III (Third Year). Pp. xiv, 171. 1935. 5s. each. (Pitman, London)

We have here the second and third parts of the book whose first part (by the same authors) was reviewed on pp. 155-6 of this volume of the *Gazette*. The method of presentation is on the same lines, but each book is intended to be complete in itself, and to provide completely for its year of the National Certificate Course. Both of these volumes under review open with a good collection of revision examples based upon the previous year's work. Almost all these examples are taken, by permission, from questions set at one or other of the public examinations.

"The authors have adopted the course of establishing the methods of instruction as far as possible by means of illustrative examples drawn chiefly from engineering practice." The electrical worker will find a special interest in the geometry chapter of Book II, where a section deals with the surface, volume and space-factor of "Former-Wound" coils, rectangular, link-shaped and elliptical.

Book II is divided by chapters into six sections, Arithmetic and Algebra, Trigonometry, Vector Quantities, Geometry and Mensuration, Algebraical Graphs and, finally, Differential Calculus.

There is, perhaps, a departure from ordinary pure mathematics in a very good chapter on Vector Quantities. Many may consider that forces, velocities, accelerations and their derivatives involving the parallelogram law, the polygon of forces and relative motion belong to the sphere of mechanics, but the work fits in here excellently and provides good examples for the use of the trigonometrical knowledge recently developed. "Engineers are very frequently concerned with vector quantities and with the solution of problems involving such quantities." In this chapter graphical methods of solution are discussed. The concluding chapter of Book III is on The Symbolic (j) Method or Complex Numbers. There the problems which are solved in Book II by graphical methods are very lucidly explained using the operator (j).

Book III, like Book II, is divided into six chapters. In addition to the one just mentioned, the sections are Algebra, Trigonometry, Graphs, Differential Calculus and Integral Calculus.

It is in this section on Graphs that the determination of physical laws is developed.

The print is very clear and the drawings are easy to follow. The authors have been very thorough in their explanations, and the three volumes make a very good set. One felt at the end that there might have been a collection of revision examples at the end of Book III similar to those on Book II which opened it. In fact one would even suggest to the authors that the examples which open Book II might also conclude Book I, and similarly those which open Book III might also conclude Book II. E. J. A.

Higher Mathematical Papers. By C. J. Cozens. Pp. iv, 176. 3s. 1935. (Arnold)

This collection of test papers is designed for use over the two years following School Certificate. Part I consists of 50 papers in Algebra, 50 in Calculus, 50 in Trigonometry and Geometry, 20 in Coordinate Geometry of the straight line and circle, and 30 miscellaneous papers. Part II has 50 papers in Algebra and Trigonometry, 30 in Calculus, 36 in Geometry (mostly the coordinate geometry of the conic sections), and 12 miscellaneous papers.

The papers, each containing four questions, have been arranged in pairs in such a way that the second is, in many cases, a reprint of the first with mere changes in the given magnitudes. The questions, mostly original, cover the non-specialist work in the VIth Form. Most users will wish for more questions in pure geometry, and there are no equations for solution by the methods of Horner or Newton. It is pleasing to find problems in probability included in both parts, and many questions ask for sketches of the functions involved.

There is a fair sprinkling of misprints, none of much consequence to the user except the confusion between match and game (No. 4 in papers 163A and 163B). W. J. L.

Mathematical Test Papers. By C. J. Cozens. Pp. 92. 1s. (Answers 6d.) 1934. (Arnold)

A useful collection of short test papers covering the school certificate syllabus in Arithmetic, Algebra, Geometry and Trigonometry. The class-room is not forgotten, for the papers are arranged in parallel sets *A* and *B*, in order that, as the author naively remarks, " the pupil's confidence in him-self may not be destroyed by his catching sight accidentally of his neighbour's answer, which may be different from his own ". E. L.

1. **Revision and Mental Tests in Arithmetic, Trigonometry, and Algebra.** By R. J. Fulford. (With answers.) Pp. 90. 1s. 1935. (University Tutorial Press)

2. **Revision and Mental Tests in Geometry.** By R. J. Fulford. Pp. 75. 1s. 1934. (University Tutorial Press)

Mathematical attainment is not to be decided by one's ability to perform incredible feats of mental gymnastics. The present generation, realising this, has relegated the calculating prodigy to the music-hall stage.

Yet the value of mental and oral work in the class-room can scarcely be over-emphasized. By means of carefully chosen questions, attractive drill can be provided in new methods and topics, and contact with previous work steadily and intimately maintained. The essential is that such questions be simple, as opposed to complex; points must be isolated, unencumbered by irrelevant ideas and difficulties. A nut is more than often destroyed, or at any rate mutilated, in the shelling of it. Anything in the nature of a *tour de force* must be avoided.

The present volumes offer excellent examples. The first contains a hundred " tests " of ten questions, each requiring about a minute for its solution. They may be used either as mental tests or as material for oral discussion, and they cover the whole range of school certificate work. In addition are provided examination papers of standard type.

In the second, the method is successfully adapted to Geometry. Questions of the " fill in the blanks " type offer excellent drill on definitions and theorems, while simple exercises, chiefly numerical, give practice in picking out salient facts from geometrical figures. Here again are a number of examination papers.

Two welcome additions to the schoolmaster's battery of small arms. E. L.

Mechanics and Hydrostatics (Elementary Science Series). By G. F. PEAKER. Pp. vii, 192. 2s. 6d. 1935. (University Tutorial Press)

We have here another book for beginners in Mechanics and Hydrostatics: it covers the usual elementary ground, including the Laws of Motion and "Centrifugal and Centripetal" forces, but not Momentum or more than the mention of Kinetic Energy. The treatment is simple, difficulties are avoided wherever possible, while most of the chapters have a few exercises and examination questions at the end; there is also an index. The book is quite interesting, and the author draws his illustrations from things of everyday life such as boats, cars and aeroplanes, while there is a large number of explanatory diagrams.

One or two details call for mention: it seems a pity that the Laws of Friction are given for limiting friction, and non-limiting friction is only brought in at the end, when so much of the friction met with in practice is non-limiting, and boys so easily get the idea that all friction is limiting. Then, in dealing with the constant acceleration formulae, the fact that s does not always represent the distance travelled should have been made clearer. Gravitational units are used throughout, and the author gets over the mass difficulty by introducing the idea of mass in Chap. 2 as "quantity of matter", with the alternative definition that "two bodies are of equal mass if their weights are equal when they are in the same neighbourhood". J. W. H.

The Calculus. By H. H. DALAKER and H. E. HARTIG. 3rd edition. Pp. viii, 276. 12s. 6d. 1935. (McGraw-Hill)

The first edition of this book was reviewed in the *Gazette* for December, 1931. Evidently it has been favourably received, but probably its success has lain in the United States rather than in this country, for its exposition is designed on lines not usually followed here. Basic concepts are brought in with so little explanatory matter that the student might easily acquire facility in working examples without having any real grasp of the subject.

The collection of examples, already good, has been considerably extended, though it is a pity that each question has its answer immediately appended; "drill" questions lose some of their value in this way. Several figures have been re-drawn, and a new figure, of the logarithmic curve, has been added; this should be re-drawn in the next edition, as it is too small and the curve is made tangent to the axis of y. T. A. A. B.

BUREAU FOR THE SOLUTION OF PROBLEMS.

THIS is under the direction of Mr. A. S. Gosset Tanner, M.A., Derby School, Derby, to whom all inquiries should be addressed, accompanied by a stamped and addressed envelope for the reply. Applicants, who must be members of the Mathematical Association, should wherever possible state the source of their problems and the names and authors of the textbooks on the subject which they possess. As a general rule the questions submitted should not be beyond the standard of University Scholarship Examinations. Whenever questions from the Cambridge Scholarship volumes are sent, it will not be necessary to copy out the question in full, but only to send the reference, *i.e.*, volume, page, and number. The names of those sending the questions will not be published.

PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD.
THE UNIVERSITY PRESS, GLASGOW

NORTH EASTERN BRANCH.

Report for 1934.

The first meeting of the year was held on Saturday, 24th February, at Palace Green, Durham. The meeting was held jointly with the Mathematics Section of the Annual Conference of University and School Teachers of Schools taking the Certificate Examinations of the University of Durham. Mr. H. A. Wheeler, of the Gateshead Secondary School, opened a discussion on " Graphical Methods ". The speaker said that we should keep in mind the Calculus rather than Coordinate Geometry. He divided the teaching into several stages: (*a*) drawing the columnar type of graph, (*b*) plotting points without drawing ordinates, (*c*) graphs of the journey type, (*d*) graphs drawn geometrically, *e.g.* ellipse, sine curve, cycloid, (*e*) interpolation, (*f*) formal algebraic functions and solution of equations by intersections. Later stages involve the obtaining a formula from a set of experimental results, and ideas of the Calculus.

The second meeting, the Annual General Meeting, was held on Saturday, 20th October, at Armstrong College. Dr. T. H. Havelock, M.A., D.Sc., F.R.S., Professor of Mathematics at Armstrong College, was elected President of the Branch for the next two years. An address was given on " Some Points of Detail in our Elementary Teaching ", by Mr. A. S. Gosset Tanner of Derby School. In his remarks, the speaker derided the idea that children " discovered " the facts of Geometry. He was not convinced that superposition was wrong, and he was still in search of a satisfactory substitute. A more careful wording of theorems was recommended, attention being called to the unsatisfactory nature of expressions such as " Between the same parallels ", " Angles in the same segment ", " Base and Height ", etc. Children, he found, seldom understood proofs by symmetry: this idea is one that should be allowed to develop by growth. The postponement of inequality theorems was advised, as was care in the treatment of the " Algebraical Theorems ", since looseness in talking leads to sloppiness in thinking. He concluded by mentioning some of the problems received by the Problems Bureau of the Association, of which he is the Honorary Secretary.

The last meeting of the year was held on Saturday, 1st December, again at Armstrong College. Mr. A. B. Oldfield, of Pudsey Technical College, opened a discussion on the Association's Report on the Teaching of Algebra. He said that there was a tendency in certain quarters to decry the teaching of Algebra, thus teachers ought to be acquainted with the reasons why it should be taught. In the objections to the teaching of mechanical operations, the pendulum had swung too far. He advocated a Mathematical Laboratory in Schools, and condemned the mechanical use of a constant in teaching Variation. In the formula approach to Algebra, the teaching of Mensuration benefits, but the pupils tend to become imbued with a distaste for the subject. The Report lacked a list of words used in Algebra together with their derivatives. In the ensuing discussion, Mr. S. H. Stelfox, H.M.I., advocated a more extensive use of logarithmic paper, and Mr. A. Fletcher showed how variation could be taught without the use of " k ", and said that if it is used, a physical meaning should be obtained for it.

At the Annual General Meeting, Miss M. Waite stated that she was unable, owing to pressure of other work, to accept nomination as Honorary (Joint) Secretary, and the Branch, accepting her resignation, expressed thanks for her services since the formation of the Branch in 1928.

J. W. Brooks, Hon. Sec.

THE LIBRARY.

160 Castle Hill, Reading.

The Librarian reports gifts as follows:

From Mr. **A. W. Siddons,** to mark his year as President:

C. Godfrey and A. W. Siddons
Teaching of Elementary Mathematics - - - - 1931

A. W. Siddons and C. T. Daltry
Elementary Algebra; II, III - - - - - 1934

A. W. Siddons and R. T. Hughes
Junior Geometry - - - - - - - - - 1930
Practical and Theoretical Geometry - - - 1926
Trigonometry - - - - - - - - - 1929

From Mr. **C. W. Adams:**

W. S. Burnside and A. W. Panton
Theory of Equations {4}; I - - - - - - 1899
Earlier editions were in one volume.

C. G. A. Harnack Introduction to the Calculus - - - - - - 1891

From Mr. **R. Goormaghtigh,** an offprint of his paper on the orthopole.

From Prof. **H. R. Hamley,** his book:
Functional Thinking in Mathematics - - - - 1934

From Mr. **P. J. Harris,** his book:
Problem Papers in Elementary Mathematics - - 1935

From Sir **T. L. Heath,** a collection of German and Italian textbooks of geometry, with

W. Killing Grundlagen der Geometrie (2 vols.) - - 1893, 1898

H. Schotten Inhalt und Methode des Planimetrischen Unterrichts (2 vols.) - - - - - - - - 1890, 1893
A comparison of methods used in a large number of textbooks of elementary geometry.

From Prof. **N. Kryloff,** a monograph on resonance, of which he and Dr. N. Bogoliùboff were joint authors.

From Mr. **W. J. Langford:**

P. S. de Laplace Exposition du Système du Monde {5} - - - - 1824

From Mr. **A. B. Oldfield:**

J. M. Whittaker Interpolatory Function Theory - Cambridge 33 1935

From Mr. **F. P. White:**

G. Loria Storia delle Matematiche; III - - - - - 1933

From Miss **R. A. Clayton** and from Prof. **A. Lodge,** collections of back numbers of the *Gazette.*

BOOKS RECEIVED FOR REVIEW.

N. Kuppuswami Aiyangar. *The teaching of mathematics in the new education.* Pp. vii, 420, v. Rs. 5. 1935. (Training College, Trivandrum, South India)

W. N. Bond. *Probability and random errors.* Pp. viii, 141. 10s. 6d. 1935. Arnold)

E. T. Copson. *An introduction to the theory of functions of a complex variable.* Pp. 448. 25s. 1935. (Oxford)

H. Crew. *The rise of modern physics.* 2nd edition. Pp. xix, 434. 18s. 1935. (Baillière, Tindall and Cox)

L. Crosland. *Higher school geometry.* Pp. xiv, 322, xx. 6s. 1935. (Macmillan)

H. T. Davis and **W. F. C. Nelson.** *Elements of statistics.* Pp. xi, 424. 17s. 6d. 1935. (Principia Press, Bloomington, Indiana ; Williams and Norgate)

H. Estève et **H. Mitault.** *Cours de Géométrie. II. Géométrie dans l'Espace.* Pp. viii, 283. 20 fr. 1936. (Gauthier-Villars)

L. N. G. Filon. *An introduction to projective geometry.* 4th edition. Pp. xviii, 407. 1935. 16s. (Arnold)

D. Hilbert. *Gesammelte Abhandlungen. III. Analysis, Grundlagen der Mathematik. Physik. Verschiedenes.* Pp. vii, 435. RM. 33.75. 1935. (Springer)

T. W. Ward Hill. *Elementary analytical geometry.* Revised edition. Pp. viii, 264. 4s. 6d. 1935. (Harrap)

L. M. Kells. *Elementary differential equations.* 2nd edition. Pp. xii, 248. 12s. 1935. (McGraw-Hill)

G. Loria. *Il Passato e il Presente delle principali Teorie Geometriche. Storia e Bibliografia.* 4th edition. Pp. xxiv, 468. 1931. (Cedam, Padoa)

G. Loria. *Storia delle Mathematiche. III. Dall'alba del secolo XVIII al tramonto del secolo XIX.* Pp. 607. L. 23. 1933. (Sten, Turin)

J. Maclean. *Graphs and statistics. Elementary applications of mathematical methods.* Pp. xv, 200. Rs. 4. 1926. (Govind, Bombay)

J. Maclean. *Descriptive mathematics.* Pp. xvi, 143. Rs. 2.8. 1935. (Macmillan)

S. Mandelbrojt. *Séries de Fourier et classes quasi-analytiques de fonctions.* Pp. viii, 156. 35 fr. 1935. (Gauthier-Villars)

G. A. Miller. *The collected works of George Abram Miller. I.* Pp. xi, 475. $7.50. 1935. (University of Illinois Press)

N. Miller. *A first course in differential equations.* Pp. 148. 7s. 6d. 1935. (Oxford)

A. H. G. Palmer and **K. S. Snell.** *Mechanics for the use of higher forms in schools, and first year students at the universities.* Pp. xiv, 335. 15s. 1935. (University of London Press)

E. A. Reeves. *Hints to travellers. I. Survey and field astronomy.* 11th edition. Pp. viii, 448. 16s. 1935. (Royal Geographical Society)

A. Ritchie-Scott. *A complete school algebra.* Pp. 711. With answers, 8s. 6d. without answers, 7s. 6d. Part I. With Answers, 5s. ; without answers, 4s. Part II. With answers, 5s. ; without answers, 4s. 1935. (Harrap)

D. K. Wilson. *The history of mathematical teaching in Scotland to the end of the eighteenth century.* Pp. viii, 99. 5s. 1935. Publications of the Scottish Council for Research in Education, 8. (University of London Press)

A. Wisdom. *Century sum books. V A, V B.* Pp. 64 each. Paper 9d., limp cloth, 11d. each. 1935. (University of London Press)

JOURNALS RECEIVED.

When no number is attached, no part has been received since a previous acknowledgment.

Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität. **9** : 3-4 ; **10** ; **11** : 1-2.

American Journal of Mathematics. **57** : 2, 3.

American Mathematical Monthly. **42** : 4, 5, 6, 7.

readily identified, and M. Lemaire's accounts of the curves unfold with the deceptive inevitability that betokens a writer who is completely master of his subject. But note the plural. The definition covers both kinds of curve, the basic distinction being one of sign alone, but M. Lemaire gives separate short chapters on the general hypocycloid and the general epicycloid, with only occasional references from the later chapter to the earlier for details of proofs; the continuity and independence of each chapter is secured at the cost not merely of a great deal of repetition but also of the omission of all parallelisms and contrasts. Similarly the sections on particular curves include paragraphs that vary in only the most trivial manner from one curve to another; the results are individual, but nothing except the results is being learned by their accumulation.

The hypocycloids with three and four cusps, the epicycloids with one, two, and three cusps, and the two degenerate cases, the cycloid and the involute of the circle, all receive some attention, but half of the book is devoted to the first of these special curves. It is well known that the envelope of the Wallace-Simson lines of any triangle is a three-cusped hypocycloid, and once this curve is accepted as a familiar figure, a new chapter is begun in the geometry of the triangle. Other interesting problems concern conics which have multiple contact with a three-cusped hypocycloid, and simple Cremona transformations connecting a three-cusped hypocycloid with a circle. It is somewhat surprising that a curve which was studied by Steiner, Laguerre, and Cremona has not been the subject of an independent treatise, and M. Lemaire's book, of which a copy has at last reached us, is welcome as coming near to filling this gap in geometrical literature. E. H. N.

Premier Livre du Tétraèdre. By P. Couderc and A. Balliccioni. Pp. viii, 204. 40 fr. 1935. (Gauthier-Villars)

A textbook on the Tetrahedron has long been overdue, and we have here a praiseworthy attempt to supply the need. As far as one can judge from this first volume (no contents list is given of the second), the standard is meant to be that of the Scholarship class in secondary schools. The various types of tetrahedron are treated in detail, and many formulae are given concerning the orthocentric and equifacial, as well as the general figure. There is also a full discussion of the related spheres, including those tangent to a skew quadrilateral. (No coordinate geometry is used in the book.)

It is evident, from the above synopsis, that the authors have been at pains to present a methodical and systematic treatment of the subject, but there is, to our mind, a serious omission in that no use is made of vector algebra; so many of the formulae become automatic when expressed in terms of scalar and vector products. If we take, for example, the edges DA, DB, DC as the vectors α, β, γ, and if P is the point whose tetrahedral coordinates are (x, y, z, t), then we have at once $\overline{DP}=x\alpha+y\beta+z\gamma$. If P' is (x', y', z', t'), then $\overline{PP'}=(x'-x)\,\alpha+(y'-y)\,\beta+(z'-z)\,\gamma$, so that expressions for PP'^2, for the perpendicularity of two lines, and so forth, can be written down at once.

Clearly much play can be made with the foregoing principles, and there is no reason why the boy in the Mathematical Sixth should find difficulty with the amount of vector lore required. For the orthocentric tetrahedron, in particular, the use of vectors not only makes the work exceedingly simple and suggestive, but enables the properties to be instantaneously extended to similar configurations in higher-dimensional space. Of course, it may be intended to adopt this point of view in the second volume. H. L.

Exercices d'Analyse. IV. Équations aux dérivées partielles du premier ordre. By G. JULIA. Pp. 230. 60 fr. 1935. (Gauthier-Villars)

In this fourth volume Professor Julia continues the general plan so admirably carried out in the earlier parts of his valuable *Exercices*. Assuming that his readers are tolerably familiar with the more elementary theory and machinery of partial differential equations, as given, for example, by Goursat, he aims partly indeed at increasing their powers of manipulation but much more at strengthening their grasp of ideas and furthering their penetration into the real meaning of the subject. It is one which beginners frequently find difficult; for in coming to it from ordinary differential equations they find the partial equations less clear-cut and much less concrete. To counteract this feeling Professor Julia, with all the expository skill we expect from him, stresses the geometrical aspect of the subject; frequently he gives two solutions to a problem, one with an analytical and the other with a geometrical bias, and by comparing the two the reader should find it much easier to comprehend the essential content lying behind the algebraic façade.

Of the 55 problems propounded and solved, most are taken either from papers in calculus at the Sorbonne or from Goursat. The whole set of volumes is of tremendous assistance both to those who are learning and to those who are teaching analysis. We look forward with pleasure for the appearance of the remaining part to complete a work of which the utility and interest cannot be too highly praised. T. A. A. B.

Esercizi di Analisi infinitesimale. By G. VIVANTI. 3rd edition. Pp. 404. L. 55. 1935. (Lattes, Turin)

Professor Vivanti's book is a valuable complement to the usual Continental textbook on the calculus. It contains 573 examples, with solutions either fully worked or, in a few cases, sketched, distributed as follows: preliminaries on limits, continuity, etc. (14); derivatives and integrals of functions of one variable (233); derivatives and integrals of functions of several variables (35); geometrical applications (99); differential equations (178); calculus of variations (14). References are given to the author's *Lezioni di analisi matematica.*

The section on linear differential equations with constant coefficients suggests that the standard way of obtaining the complete solution is by "variation of parameters", which seems rather ponderous to be applied, for example, to No. 504:

$$y''' - 3y' + 2y = x^2.$$

In this country, however, we are perhaps too contemptuous of this method. It is good to see that in dealing with partial differential equations a fair show is given to equations in more than two independent variables.

A student reading this book should find both his manipulative powers and his respect for fine points increased. T. A. A. B.

Höhere Mathematik für Mathematiker, Physiker und Ingenieure. III. By R. ROTHE. Pp. ix, 238. RM. 6.60. 1935. Teubners mathematische Leitfäden, 23. (Teubner)

This volume completes the text of Dr. Rothe's manual, the fourth volume being devoted to further examples and solutions.

Two sections of this volume deal with curvilinear coordinates, and curvilinear and multiple integrals. The remaining section, filling roughly half the book, is on differential equations. The general plan of cementing theory and practice with a large and interesting collection of illustrative examples is

carried on throughout the book, with results which are, we think, particularly notable when we come to the differential equations. Here, although only 114 pages altogether are available, we have 30 pages of introductory matter before methods of integration are mentioned. In these 30 pages, the construction and importance of the subject are exhibited by illustrations from dynamics, physics, ballistics, electrical theory, " looping the loop ", and geometry ; but this does not mean that the author thereby considers himself exempt from the duty of dealing with the abstract theory. No sooner has he discussed the elementary methods of integration applicable to equations of the first order than he turns to the existence theorems, and gives a simple and lucid account which ought to be intelligible to anyone capable of understanding what the problem really is. Linear equations are treated concisely but adequately, save that the only method employed in the determination of particular integrals is that of variation of parameters. With a little care in presentation this method can be made to appear very natural, but to use it on quite simple constant-coefficient equations is as heavy as the exclusive employment of operational methods can be. Whatever may be true of the teaching of mathematics for technical applications, the mathematician in the making must have training in the selection of appropriate tools.

We congratulate Dr. Rothe on the completion of the main part of his treatise. Its clarity of expression and wealth of illustration should make it a valuable aid to students and teachers, whether their interests are theoretical or practical. Considering the neat format and excellent paper and printing, as well as the amazingly large amount of material included, the book is splendid value for the money. T. A. A. B.

A Complete School Algebra. By A. Ritchie-Scott. Pp. 711, including answers, 8s. 6d. ; without answers, 7s. 6d. Part I. With answers, 5s. ; without answers, 4s. Part II. With answers, 5s. ; without answers, 4s. 1935. (Harrap)

This is an arresting work, at once stimulating and perplexing.

It is offered to the world with the benediction of Professor E. T. Whittaker. That commands immediate respect ; but as Professor Whittaker is careful to say that his appreciation is that of a university professor, the schoolmaster may, without undue diffidence, criticise the suitability of the exposition for young pupils.

As to the title, the word " complete " must not be taken literally. Except for a chapter on the Multinomial theorem with some reference to partitions, the book closes with the Binomial theorem for a positive index. And as a school algebra it presents unusual features. In delaying the simple equation to Chapter VIII, it seems to react to a procedure now generally discredited. But an examination of the earlier chapters will show that this is only apparent ; indeed the author introduces much original matter and method. He looks for a maturity and responsiveness in young pupils that schoolmasters would be cheered to find. And Chapter VII preceding Chapter VIII looks like a very large cart pushing a diminutive horse.

Even so, this may represent the author's experience tried out on average beginners. If so, one would welcome an assurance to that effect and, stimulated by a courageous example, run one's teaching on bolder lines.

The book's scholarship is everywhere apparent, and offers many interesting ideas rare in elementary instruction.

Again, while the author in places seems to aim at widening the pupil's outlook and stimulating his powers, in others he elaborates a topic with an

unusual intensity of purpose. For example, the discussion of the quadratic function and its graphical treatment seem to aim at such a thorough mastery of

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

as is only needed by scholarship pupils doing analytical geometry.

Some mention of particulars of earlier chapters will exemplify these remarks.

The first batch of exercises asks the pupil to translate into formulae statements of various laws, such as those for electromotive force, Young's modulus, etc.

Chapter III, which deals with surds, includes the approach to $\sqrt{7}$ through the convergences 2, 2·6, 2·64, ... to 2·6457513 and 3, 2·7, 2·65, ... to 2·6457514. This is followed by a graphical representation of incommensurable numbers. And in the same chapter, the pupil sees

$$\frac{2\sqrt{(3\cdot1)}}{2\sqrt{(2\cdot4)}} \text{ simplified to } \frac{\sqrt{(7\cdot74)}}{1\cdot6}.$$

Chapter IV introduces graphs through linear arrangements and scales.

Chapter V on multiplication and factorisation devotes a number of pages to the identity $(a+b)(a-b)\equiv a^2-b^2$ and concludes with the question

$$\text{“ Simplify } \frac{(a-b+c)^2-(a-b-c)^2}{4(x+y)} \div \frac{a(b+c)}{x-y}.\text{”}$$

In Chapter VII the pupil is solving equations such as

$$\frac{1}{x}=\frac{1}{u-v}+\frac{1}{u+v},$$

and problems such as

“ A train runs between A and B, a distance of s miles. The scheduled time for the journey is t hours, but in going from A to B it takes h hours longer than this, making up for the loss on the return journey by going from B to A in h hours less than the scheduled time. (i) Find the average value of the two actual velocities. (ii) Find the average velocity of the train over the double journey.

Explain and reconcile the two answers.”

Then follows Chapter VIII entitled “ Simple Equations ”. And the word “ simple ” here acquires an added significance.

It is difficult to imagine that this can be a suitable book for the lower classes of schools, but the teacher who consults it will be well repaid for the perusal.

F. C. B.

A School Certificate Algebra. By W. G. Borchardt. With answers. Pp. xii, 431, lxxvi. 4s. 1935. (Rivingtons)

Borchardt's textbooks have been widely used and appreciated for more than thirty years. A popularity of this kind may imply that in spite of their acknowledged merits they are now behind the times. The present work is free from that reproach. The author explains in the Preface that he has kept an eye on the latest report of our Association, from which he has adopted many suggestions, and that much new matter has been introduced.

Chapter I introduces Algebra as generalised arithmetic, and Chapters IV and V deal with the first Four Rules, powers, roots, H.C.F. and L.C.M. from this point of view.

Chapter II deals with formulae, and Chapter III with the simple equation.

Thus in 32 pages the pupil becomes acquainted with the threefold basis of the subject.

Part I proceeds thereafter on what are now regarded as normal lines as far

as simultaneous equations of the 1st degree in three unknowns and of the 2nd in two unknowns and the Remainder Theorem.

Part II includes Progressions with a section on Annuities, Theory of Quadratics, Theory of Logarithms, Variation, Partial Fractions and Graphs. The graphical section has a technical bias and deals with gradient, the area under the graph and the law of the graph in such a case as $yx^n = c$.

Examples are numerous and varied, and there are frequent batches of revision examples. There are also XCIV Test Papers.

The requirements of various certificate examinations are more than amply met. F. C. B.

Elementary Practical Mathematics. By E. W. Golding and H. G. Green. II (Second Year). Pp. xii, 188. 1934. III (Third Year). Pp. xiv, 171. 1935. 5s. each. (Pitman, London)

We have here the second and third parts of the book whose first part (by the same authors) was reviewed on pp. 155-6 of this volume of the *Gazette*. The method of presentation is on the same lines, but each book is intended to be complete in itself, and to provide completely for its year of the National Certificate Course. Both of these volumes under review open with a good collection of revision examples based upon the previous year's work. Almost all these examples are taken, by permission, from questions set at one or other of the public examinations.

" The authors have adopted the course of establishing the methods of instruction as far as possible by means of illustrative examples drawn chiefly from engineering practice." The electrical worker will find a special interest in the geometry chapter of Book II, where a section deals with the surface, volume and space-factor of " Former-Wound " coils, rectangular, link-shaped and elliptical.

Book II is divided by chapters into six sections, Arithmetic and Algebra, Trigonometry, Vector Quantities, Geometry and Mensuration, Algebraical Graphs and, finally, Differential Calculus.

There is, perhaps, a departure from ordinary pure mathematics in a very good chapter on Vector Quantities. Many may consider that forces, velocities, accelerations and their derivatives involving the parallelogram law, the polygon of forces and relative motion belong to the sphere of mechanics, but the work fits in here excellently and provides good examples for the use of the trigonometrical knowledge recently developed. " Engineers are very frequently concerned with vector quantities and with the solution of problems involving such quantities." In this chapter graphical methods of solution are discussed. The concluding chapter of Book III is on The Symbolic (j) Method or Complex Numbers. There the problems which are solved in Book II by graphical methods are very lucidly explained using the operator (j).

Book III, like Book II, is divided into six chapters. In addition to the one just mentioned, the sections are Algebra, Trigonometry, Graphs, Differential Calculus and Integral Calculus.

It is in this section on Graphs that the determination of physical laws is developed.

The print is very clear and the drawings are easy to follow. The authors have been very thorough in their explanations, and the three volumes make a very good set. One felt at the end that there might have been a collection of revision examples at the end of Book III similar to those on Book II which opened it. In fact one would even suggest to the authors that the examples which open Book II might also conclude Book I, and similarly those which open Book III might also conclude Book II. E. J. A.

Higher Mathematical Papers. By C. J. Cozens. Pp. iv, 176. 3s. 1935. (Arnold)

This collection of test papers is designed for use over the two years following School Certificate. Part I consists of 50 papers in Algebra, 50 in Calculus, 50 in Trigonometry and Geometry, 20 in Coordinate Geometry of the straight line and circle, and 30 miscellaneous papers. Part II has 50 papers in Algebra and Trigonometry, 30 in Calculus, 36 in Geometry (mostly the coordinate geometry of the conic sections), and 12 miscellaneous papers.

The papers, each containing four questions, have been arranged in pairs in such a way that the second is, in many cases, a reprint of the first with mere changes in the given magnitudes. The questions, mostly original, cover the non-specialist work in the VIth Form. Most users will wish for more questions in pure geometry, and there are no equations for solution by the methods of Horner or Newton. It is pleasing to find problems in probability included in both parts, and many questions ask for sketches of the functions involved.

There is a fair sprinkling of misprints, none of much consequence to the user except the confusion between match and game (No. 4 in papers 163A and 163B). W. J. L.

Mathematical Test Papers. By C. J. Cozens. Pp. 92. 1s. (Answers 6d.) 1934. (Arnold)

A useful collection of short test papers covering the school certificate syllabus in Arithmetic, Algebra, Geometry and Trigonometry. The class-room is not forgotten, for the papers are arranged in parallel sets *A* and *B*, in order that, as the author naively remarks, " the pupil's confidence in himself may not be destroyed by his catching sight accidentally of his neighbour's answer, which may be different from his own ". E. L.

1. **Revision and Mental Tests in Arithmetic, Trigonometry, and Algebra.** By R. J. Fulford. (With answers.) Pp. 90. 1s. 1935. (University Tutorial Press)

2. **Revision and Mental Tests in Geometry.** By R. J. Fulford. Pp. 75. 1s. 1934. (University Tutorial Press)

Mathematical attainment is not to be decided by one's ability to perform incredible feats of mental gymnastics. The present generation, realising this, has relegated the calculating prodigy to the music-hall stage.

Yet the value of mental and oral work in the class-room can scarcely be over-emphasized. By means of carefully chosen questions, attractive drill can be provided in new methods and topics, and contact with previous work steadily and intimately maintained. The essential is that such questions be simple, as opposed to complex; points must be isolated, unencumbered by irrelevant ideas and difficulties. A nut is more than often destroyed, or at any rate mutilated, in the shelling of it. Anything in the nature of a *tour de force* must be avoided.

The present volumes offer excellent examples. The first contains a hundred " tests " of ten questions, each requiring about a minute for its solution. They may be used either as mental tests or as material for oral discussion, and they cover the whole range of school certificate work. In addition are provided examination papers of standard type.

In the second, the method is successfully adapted to Geometry. Questions of the " fill in the blanks " type offer excellent drill on definitions and theorems, while simple exercises, chiefly numerical, give practice in picking out salient facts from geometrical figures. Here again are a number of examination papers.

Two welcome additions to the schoolmaster's battery of small arms. E. L.

Mechanics and Hydrostatics (Elementary Science Series). By G. F. PEAKER. Pp. vii, 192. 2s. 6d. 1935. (University Tutorial Press)

We have here another book for beginners in Mechanics and Hydrostatics: it covers the usual elementary ground, including the Laws of Motion and "Centrifugal and Centripetal" forces, but not Momentum or more than the mention of Kinetic Energy. The treatment is simple, difficulties are avoided wherever possible, while most of the chapters have a few exercises and examination questions at the end; there is also an index. The book is quite interesting, and the author draws his illustrations from things of everyday life such as boats, cars and aeroplanes, while there is a large number of explanatory diagrams.

One or two details call for mention: it seems a pity that the Laws of Friction are given for limiting friction, and non-limiting friction is only brought in at the end, when so much of the friction met with in practice is non-limiting, and boys so easily get the idea that all friction is limiting. Then, in dealing with the constant acceleration formulae, the fact that s does not always represent the distance travelled should have been made clearer. Gravitational units are used throughout, and the author gets over the mass difficulty by introducing the idea of mass in Chap. 2 as "quantity of matter", with the alternative definition that "two bodies are of equal mass if their weights are equal when they are in the same neighbourhood". J. W. H.

The Calculus. By H. H. DALAKER and H. E. HARTIG. 3rd edition. Pp. viii, 276. 12s. 6d. 1935. (McGraw-Hill)

The first edition of this book was reviewed in the *Gazette* for December, 1931. Evidently it has been favourably received, but probably its success has lain in the United States rather than in this country, for its exposition is designed on lines not usually followed here. Basic concepts are brought in with so little explanatory matter that the student might easily acquire facility in working examples without having any real grasp of the subject.

The collection of examples, already good, has been considerably extended, though it is a pity that each question has its answer immediately appended; "drill" questions lose some of their value in this way. Several figures have been re-drawn, and a new figure, of the logarithmic curve, has been added; this should be re-drawn in the next edition, as it is too small and the curve is made tangent to the axis of y. T. A. A. B.

BUREAU FOR THE SOLUTION OF PROBLEMS.

THIS is under the direction of Mr. A. S. Gosset Tanner, M.A., Derby School, Derby, to whom all inquiries should be addressed, accompanied by a stamped and addressed envelope for the reply. Applicants, who must be members of the Mathematical Association, should wherever possible state the source of their problems and the names and authors of the textbooks on the subject which they possess. As a general rule the questions submitted should not be beyond the standard of University Scholarship Examinations. Whenever questions from the Cambridge Scholarship volumes are sent, it will not be necessary to copy out the question in full, but only to send the reference, *i.e.*, volume, page, and number. The names of those sending the questions will not be published.

PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD.
THE UNIVERSITY PRESS, GLASGOW

NORTH EASTERN BRANCH.

Report for 1934.

The first meeting of the year was held on Saturday, 24th February, at Palace Green, Durham. The meeting was held jointly with the Mathematics Section of the Annual Conference of University and School Teachers of Schools taking the Certificate Examinations of the University of Durham. Mr. H. A. Wheeler, of the Gateshead Secondary School, opened a discussion on " Graphical Methods ". The speaker said that we should keep in mind the Calculus rather than Coordinate Geometry. He divided the teaching into several stages: (*a*) drawing the columnar type of graph, (*b*) plotting points without drawing ordinates, (*c*) graphs of the journey type, (*d*) graphs drawn geometrically, *e.g.* ellipse, sine curve, cycloid, (*e*) interpolation, (*f*) formal algebraic functions and solution of equations by intersections. Later stages involve the obtaining a formula from a set of experimental results, and ideas of the Calculus.

The second meeting, the Annual General Meeting, was held on Saturday, 20th October, at Armstrong College. Dr. T. H. Havelock, M.A., D.Sc., F.R.S., Professor of Mathematics at Armstrong College, was elected President of the Branch for the next two years. An address was given on " Some Points of Detail in our Elementary Teaching ", by Mr. A. S. Gosset Tanner of Derby School. In his remarks, the speaker derided the idea that children " discovered " the facts of Geometry. He was not convinced that superposition was wrong, and he was still in search of a satisfactory substitute. A more careful wording of theorems was recommended, attention being called to the unsatisfactory nature of expressions such as " Between the same parallels ", " Angles in the same segment ", " Base and Height ", etc. Children, he found, seldom understood proofs by symmetry: this idea is one that should be allowed to develop by growth. The postponement of inequality theorems was advised, as was care in the treatment of the " Algebraical Theorems ", since looseness in talking leads to sloppiness in thinking. He concluded by mentioning some of the problems received by the Problems Bureau of the Association, of which he is the Honorary Secretary.

The last meeting of the year was held on Saturday, 1st December, again at Armstrong College. Mr. A. B. Oldfield, of Pudsey Technical College, opened a discussion on the Association's Report on the Teaching of Algebra. He said that there was a tendency in certain quarters to decry the teaching of Algebra, thus teachers ought to be acquainted with the reasons why it should be taught. In the objections to the teaching of mechanical operations, the pendulum had swung too far. He advocated a Mathematical Laboratory in Schools, and condemned the mechanical use of a constant in teaching Variation. In the formula approach to Algebra, the teaching of Mensuration benefits, but the pupils tend to become imbued with a distaste for the subject. The Report lacked a list of words used in Algebra together with their derivatives. In the ensuing discussion, Mr. S. H. Stelfox, H.M.I., advocated a more extensive use of logarithmic paper, and Mr. A. Fletcher showed how variation could be taught without the use of " k ", and said that if it is used, a physical meaning should be obtained for it.

At the Annual General Meeting, Miss M. Waite stated that she was unable, owing to pressure of other work, to accept nomination as Honorary (Joint) Secretary, and the Branch, accepting her resignation, expressed thanks for her services since the formation of the Branch in 1928.

J. W. Brooks, Hon. Sec.

THE LIBRARY.

160 Castle Hill, Reading.

The Librarian reports gifts as follows:

From Mr. **A. W. Siddons,** to mark his year as President:

C. Godfrey and A. W. Siddons
Teaching of Elementary Mathematics - - - - 1931

A. W. Siddons and C. T. Daltry
Elementary Algebra; II, III - - - - - 1934

A. W. Siddons and R. T. Hughes
Junior Geometry - - - - - - - - - 1930
Practical and Theoretical Geometry - - - 1926
Trigonometry - - - - - - - - - 1929

From Mr. **C. W. Adams**:

W. S. Burnside and A. W. Panton
Theory of Equations {4}; I - - - - - - 1899
Earlier editions were in one volume.

C. G. A. Harnack Introduction to the Calculus - - - - - 1891

From Mr. **R. Goormaghtigh,** an offprint of his paper on the orthopole.

From Prof. **H. R. Hamley,** his book:
Functional Thinking in Mathematics - - - 1934

From Mr. **P. J. Harris,** his book:
Problem Papers in Elementary Mathematics - - 1935

From Sir **T. L. Heath,** a collection of German and Italian textbooks of geometry, with

W. Killing Grundlagen der Geometrie (2 vols.) - - 1893, 1898

H. Schotten Inhalt und Methode des Planimetrischen Unterrichts (2 vols.) - - - - - - - 1890, 1893
A comparison of methods used in a large number of text-books of elementary geometry.

From Prof. **N. Kryloff,** a monograph on resonance, of which he and Dr. N. Bogoliùboff were joint authors.

From Mr. **W. J. Langford**:

P. S. de Laplace Exposition du Système du Monde {5} - - - 1824

From Mr. **A. B. Oldfield**:

J. M. Whittaker Interpolatory Function Theory - Cambridge 33 1935

From Mr. **F. P. White**:

G. Loria Storia delle Matematiche; III - - - - - 1933

From Miss **R. A. Clayton** and from Prof. **A. Lodge,** collections of back numbers of the *Gazette*.

BOOKS RECEIVED FOR REVIEW.

N. Kuppuswami Aiyangar. *The teaching of mathematics in the new education.* Pp. vii, 420, v. Rs. 5. 1935. (Training College, Trivandrum, South India)

W. N. Bond. *Probability and random errors.* Pp. viii, 141. 10s. 6d. 1935. Arnold)

E. T. Copson. *An introduction to the theory of functions of a complex variable.* Pp. 448. 25s. 1935. (Oxford)

H. Crew. *The rise of modern physics.* 2nd edition. Pp. xix, 434. 18s. 1935. (Baillière, Tindall and Cox)

L. Crosland. *Higher school geometry.* Pp. xiv, 322, xx. 6s. 1935. (Macmillan)

H. T. Davis and **W. F. C. Nelson.** *Elements of statistics.* Pp. xi, 424. 17s. 6d. 1935. (Principia Press, Bloomington, Indiana ; Williams and Norgate)

H. Estève et **H. Mitault.** *Cours de Géométrie. II. Géométrie dans l'Espace.* Pp. viii, 283. 20 fr. 1936. (Gauthier-Villars)

L. N. G. Filon. *An introduction to projective geometry.* 4th edition. Pp. xviii, 407. 1935. 16s. (Arnold)

D. Hilbert. *Gesammelte Abhandlungen. III. Analysis, Grundlagen der Mathematik. Physik. Verschiedenes.* Pp. vii, 435. RM. 33.75. 1935. (Springer)

T. W. Ward Hill. *Elementary analytical geometry.* Revised edition. Pp. viii, 264. 4s. 6d. 1935. (Harrap)

L. M. Kells. *Elementary differential equations.* 2nd edition. Pp. xii, 248. 12s. 1935. (McGraw-Hill)

G. Loria. *Il Passato e il Presente delle principali Teorie Geometriche. Storia e Bibliografia.* 4th edition. Pp. xxiv, 468. 1931. (Cedam, Padoa)

G. Loria. *Storia delle Mathematiche. III. Dall'alba del secolo XVIII al tramonto del secolo XIX.* Pp. 607. L. 23. 1933. (Sten, Turin)

J. Maclean. *Graphs and statistics. Elementary applications of mathematical methods.* Pp. xv, 200. Rs. 4. 1926. (Govind, Bombay)

J. Maclean. *Descriptive mathematics.* Pp. xvi, 143. Rs. 2.8. 1935. (Macmillan)

S. Mandelbrojt. *Séries de Fourier et classes quasi-analytiques de fonctions.* Pp. viii, 156. 35 fr. 1935. (Gauthier-Villars)

G. A. Miller. *The collected works of George Abram Miller. I.* Pp. xi, 475. $7.50. 1935. (University of Illinois Press)

N. Miller. *A first course in differential equations.* Pp. 148. 7s. 6d. 1935. (Oxford)

A. H. G. Palmer and **K. S. Snell.** *Mechanics for the use of higher forms in schools, and first year students at the universities.* Pp. xiv, 335. 15s. 1935. (University of London Press)

E. A. Reeves. *Hints to travellers. I. Survey and field astronomy.* 11th edition. Pp. viii, 448. 16s. 1935. (Royal Geographical Society)

A. Ritchie-Scott. *A complete school algebra.* Pp. 711. With answers, 8s. 6d. without answers, 7s. 6d. Part I. With Answers, 5s. ; without answers, 4s. Part II. With answers, 5s. ; without answers, 4s. 1935. (Harrap)

D. K. Wilson. *The history of mathematical teaching in Scotland to the end of the eighteenth century.* Pp. viii, 99. 5s. 1935. Publications of the Scottish Council for Research in Education, 8. (University of London Press)

A. Wisdom. *Century sum books. V A, V B.* Pp. 64 each. Paper 9d., limp cloth, 11d. each. 1935. (University of London Press)

JOURNALS RECEIVED.

When no number is attached, no part has been received since a previous acknowledgment.

Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität. 9 : 3-4 ; 10 ; 11 : 1-2.

American Journal of Mathematics. 57 : 2, 3.

American Mathematical Monthly. 42 : 4, 5, 6, 7.

Anales de la Sociedad Científica Argentina. 118 : 4, 5 ; 119 : 1-3, 4.
Annales de la Société Polonaise de Mathématique. 12.
Annali della R. Scuola di Pisa. Ser. 2. 4 : 3, 4.
Annals of Mathematics. 36 : 2, 3.
Anuario (Univ. Nac. de la Plata).
Berichte über die Verhandlungen der Akad. der Wiss. zu Leipzig : Math.-Phys. Klasse. 86 : 5 ; 87 : 1.
Boletin Matematico. 7. 8 : 1, 2, 3, 4, 5, 6, 7.
Boletin Matematico Elemental. 5 : 1, 2, 3, 4.
Boletín del Seminario Matemático Argentino. 4 : 15, 16, 17.
Bollettino della Unione Matematica Italiana. 14 : 2, 3, 4.
Bulletin de l'Académie Royale Serbe. A. 2.
Bulletin of the American Mathematical Society. 41 : 2, 3, 4, 5, 6, 7, 8, 9.
Bulletin of the Calcutta Mathematical Society. 26 : 2.
Communications . . . de Kharkoff. Ser. 4. 11.
Contribución al Estudio de las Ciencias Físicas y Matemáticas.
L'Enseignement Mathématique. 33 : 3, 4 ; 34 : 1-2.
Esercitazioni Matematiche. Ser. 2. 8 : 4-5, 6-7, 8-9.
Gazeta Matematica. 40 : 8, 9, 10, 11, 12 ; 41 : 1, 2.
Half-Yearly Journal of the Mysore University.
Jahresbericht der Deutschen Mathematiker-Vereinigung. 45 : 1-4, 5-8.
Japanese Journal of Mathematics. 11 : 4 ; 12 : 1, 2.
Journal of the Faculty of Sciences, Hokkaido. Ser. 1. 2 : 4 ; 3 : 1, 2.
Journal of the Indian Mathematical Society. N.S. 1 : 4, 5, 6.
Journal of the London Mathematical Society. 10 : 2, 3.
Journal of the Mathematical Association of Japan. 17 : 2, 3, 4.
Matemática Elemental. 4 : 1-2.
Mathematical Notes. 29.
Mathematics Student. 2 : 4 ; 3 : 1, 2.
Mathematics Teacher. 28 : 4, 5, 6.
Monatshefte für Mathematik und Physik. 39 : 1.
Nieuw Archief voor Wiskunde. Ser. 2. 18 : 3.
Periodico di Matematiche.
Proceedings of the Edinburgh Mathematical Society. Ser. 2. 4 : 3.
Proceedings of the Physico-Mathematical Society of Japan. Ser. 3. 17 : 3, 4, 5, 6, 7, 8, 9.
Publicaciones . . . . Físico-Matemáticas . . . . de la Plata. 79.
Publications de la Faculté des Sciences de Masaryk. 198, 205.
Revista de Ciencias (Peru).
Revista Matemática Hispano-Americana (Madrid). Ser. 2. 10 : 3-4, 5, 6, 7.
Revue Semestrielle des Publications Mathématiques. 39 : 6.
School Science and Mathematics. 35 : 4, 5, 6, 7.
Science Progress. 117, 118.
Scripta Mathematica. 3 : 2, 3.
Sitzungsberichte der Berliner Mathematischen Gesellschaft.
Studia Mathematica.
Unterrichtsblätter für Mathematik und Naturwissenschaften. 41 : 4, 5, 6, 7, 8.
Wiskundige Opgaven met de Oplossingen. 16 : 3.